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The Tauberian Theorems which Interrelate Different Riesz Means

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INTRODUCTION

Let (λ_1, κ_1), (λ_2, κ_2), (λ_3, κ_3) be (any) three Riesz-means, and consider all functions which are transformed by (λ_1, κ_1), (λ_2, κ_2) into functions whose rate of increase does not exceed some given orders, e.g., let[1]

$$A_{\lambda_1}^{\kappa_1}(x) \preccurlyeq V_1(x), \qquad A_{\lambda_2}^{\kappa_2}(x) \preccurlyeq V_2(x). \tag{1}$$

Then the question arises, and the discussion and solution of this question is the main purpose of this paper, about the existence and determination of the best possible consequence of (1) for the (λ_3, κ_3)-transform; in other words we want to find the "minimal" V_3 such that

$$A_{\lambda_3}^{\kappa_3}(x) \preccurlyeq V_3(x), \tag{2}$$

is a consequence of (1)[2].

Several theorems of this type for special constellations of the means (λ_i, κ_i) are known, and it is customary to divide them into Abelian and Tauberian theorems depending on whether (2) follows from one of the assumptions alone[3] (like the theorems of consistency) or not (like the convexity theorem).

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[1] Throughout this paper we will assume that order functions like V_1, V_2 and the sequences λ_i are of logarithmic–exponential type, and we find it convenient to use the notations $\prec$, $\preccurlyeq$, $\asymp$, $\underset{\asymp}{=}$, $\sim$ (see [2]) which are natural in connection with such functions. In what follows, logarithmic-exponential functions will be called L-functions, and $f \in L$ means that f is an L-function for large values of the argument.

[2] This problem is of "O-type". We will also discuss the corresponding "o-problems", and problems of "mixed" type.

[3] We do not exclude the case $(\lambda_1, \kappa_1) = (\lambda_2, \kappa_2)$.

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But these theorems do not cover all possible constellations, and we shall prove some new ones (essentially a Tauberian theorem). It turns out that suitable combinations of two Abelian and one Tauberian theorem always lead from (1) to the best possible (2), if (roughly speaking) only the λ's and V's are smooth enough, if the V's do not decrease or increase too fast, and if the orders are in [0, 1] (a restriction which can probably be omitted).

Survey of Results

Prior to the discussion of the structure of the Abelian and Tauberian theorems we give the definition of the functions $A_\lambda^\kappa(x)$ which is used here (our definition corresponds to $\kappa A_\lambda^\kappa(\lambda(x))$ in the notation of [1]).

Suppose that

$$\lambda \in C_1[0, \infty), \quad \lambda \in L, \quad \lambda(0) = 0, \quad \lambda'(x) > 0, \quad \lambda(x) \to \infty, \tag{3}$$

and that

$$A \in M, \qquad \text{i.e.,} \qquad A \in L_\infty(0, r) \qquad \text{for every} \quad r > 0,$$

or

$$A \in S, \quad \text{i.e.,} \quad A(t) = \sum_{0 \leqslant \nu < t} a_\nu \qquad (t \geqslant 0).$$

Then we define[4]

$$A_\lambda^\kappa(x) = \int_0^x (\lambda(x) - \lambda(t))^{\kappa-1} \lambda'(t) A(t)\, dt, \quad \kappa > 0; \qquad A_\lambda^0(x) = A(x),$$

and A is called summable (λ, κ) to s if $(\kappa/\lambda^\kappa(x))\, A_\lambda^\kappa(x) \to s$ as $x \to \infty$. For functions $\lambda \in L$ we will write $\Lambda(x) = \lambda(x)/\lambda'(x)$ (λ may have subscripts, etc., which will also appear with the corresponding Λ). Since the detailed formulation of our results turns out to be rather complicated, it seems appropriate to discuss the main aspects in a simplified form, which exhibits more clearly the various interrelations.

From the viewpoint of summability our first Abelian Theorem leads from (λ_1, κ_1) to stronger methods (λ_3, κ_3), i.e., it is of the consistency type (denoted by C). In that case the limitation order can only increase while the corresponding Tauberian condition can only become stronger, i.e.,

$$\Lambda_3^{\kappa_3} \geqslant \Lambda_1^{\kappa_1}, \qquad \Lambda_3 \geqslant \Lambda_1 .$$

The remaining Abelian theorems are of the limitation type.

[4] At this point we emphasize that in this paper functions A_λ^κ are considered only when λ satisfies (3).

Technically, the latter theorems can be divided into two categories depending on whether $\kappa_3 \leqslant \kappa_1$ or $\kappa_3 > \kappa_1$. Theorems of the first category will be denoted by L, and theorems of the second category can be obtained as a combination of theorems L and C, hence we will denote them by LC. In a simplified form[5] these theorems can be formulated as follows: Suppose that $A_{\lambda_1}^{\kappa_1} \preccurlyeq V_1$, and that $\Lambda_1 \succcurlyeq 1$, $\Lambda_3 \succcurlyeq 1$. Then

$$A_{\lambda_3}^{\kappa_3} \preccurlyeq \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \quad \text{if } \Lambda_3 \succcurlyeq \Lambda_1 \quad \text{and} \quad \Lambda_1^{\kappa_1} \preccurlyeq \Lambda_3^{\kappa_3}, \tag{C}$$

$$A_{\lambda_3}^{\kappa_3} \preccurlyeq \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \frac{\Lambda_1^{\kappa_1}}{\Lambda_3^{\kappa_3}} \quad \text{if } \kappa_3 \leqslant \kappa_1 \quad \text{and} \quad \Lambda_1^{\kappa_1} \succcurlyeq \Lambda_3^{\kappa_3}, \tag{L}$$

$$A_{\lambda_3}^{\kappa_3} \preccurlyeq \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_1} \quad \text{if } \kappa_3 \geqslant \kappa_1 \quad \text{and} \quad \Lambda_3 \preccurlyeq \Lambda_1 .^6 \tag{LC}$$

The logical structure of these theorems can be illustrated as follows. Let the points on the horizontal axis of a coordinate system "correspond" to the functions λ (such that $<$ and $\prec$ are consistent), and take the vertical axis as κ-axis. Then the means (λ, κ) "correspond" to points in the plane, and the Abelian Theorems are indicated by arrows in the following diagram[7]

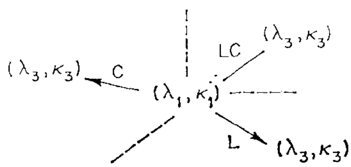

DIAGRAM 1.

The broken lines divide the regions of validity of the theorems. The line dividing C and L may be horizontal (e.g., if $\Lambda_1(x) = x$) or vertical (e.g., if $\Lambda_1(x) = 1$). Observe, that in the "region" C we have the same average order $V_1/\lambda_1^{\kappa_1}$, and that in the "region" L we have the same limitation order $(V_1/\lambda_1^{\kappa_1})\,\Lambda_1^{\kappa_1}$.

[5] The simplifications are essentially the following ones. We consider only functions $A \in S$, and we replace integrals like $\int_0^x f(t)\,dt$ by $xf(x)$.

[6] Theorem (LC) is obviously a combination of Theorems C and L (use L first to obtain an estimate of $A_{\lambda_3}^{\kappa_1}$, and then apply C to obtain the estimate of LC. All three Abelian Theorems can be condensed into a single one: $A_{\lambda_3}^{\kappa_3} \preccurlyeq \lambda_3^{\kappa_3}(V_1/\lambda_1^{\kappa_1})(1 + \Lambda_1^{\kappa_1}/\Lambda_3^{\kappa_3} + (\Lambda_1/\Lambda_3)^{\kappa_1})$.

[7] Relations $\Lambda^* \asymp \Lambda$ resp. $\Lambda^* \preccurlyeq \Lambda$ are equivalent to $\lambda^\alpha \preccurlyeq \lambda^* \preccurlyeq \lambda^\beta$ (for some constants $0 < \alpha < \beta$) resp. $\lambda^* \succcurlyeq \lambda^\delta$ for some constant $\delta > 0$ (see, e.g., [2, Theorem 23]). Hence, in our diagram, larger λ's correspond to smaller Λ's. In this diagram we assume that methods (λ, κ), (λ^*, κ) with $\Lambda \asymp \Lambda^*$ (such methods are equivalent in summability) are represented by the same point.

Next, we discuss the Tauberian theorem (denoted by T) in a simplified form. It improves the conclusion of L whenever $\Lambda_1 \leqslant \Lambda_3$. Starting from the assumption $A_{\lambda_1}^{\kappa_1} \leqslant V_1$ it leads under a Tauberian condition to conclusions $A_{\lambda_3}^{\kappa_3} \leqslant \lambda_3^{\kappa_3}(V_1/\lambda_1^{\kappa_1})V$, $1 \leqslant V \leqslant \Lambda_1^{\kappa_1}/\Lambda_3^{\kappa_3}$ (the Tauberian condition depends on V), i.e., in the region $\Lambda_1 \leqslant \Lambda_3$, $\Lambda_1^{\kappa_1} \geqslant \Lambda_3^{\kappa_3}$, $\kappa_3 < \kappa_1$, it interpolates between the orders of $A_{\lambda_3}^{\kappa_3}$ appearing in C and L (and, in particular, for $V \asymp 1$ it extends the conclusion of C to this region). The Tauberian condition is $A_{\lambda_2}^{\kappa_3} \leqslant V_2$, $\Lambda_2 \leqslant \Lambda_1$ where V_2 and λ_2 are determined by the following requirements:

(i) the "L-consequence" of $A_{\lambda_3}^{\kappa_3} \leqslant \lambda_3^{\kappa_3}(V_1/\lambda_1^{\kappa_1})V$ is $A_{\lambda_2}^{\kappa_3} \leqslant V_2$, and

(ii) the "C-consequence" $A_{\lambda_2}^{\kappa_1} \leqslant V^*$ of $A_{\lambda_2}^{\kappa_3} \leqslant V_2$, and the "$L$-consequence" $A_{\lambda_2}^{\kappa_1} \leqslant V^{**}$ of $A_{\lambda_1}^{\kappa_1} \leqslant V_1$ are equivalent, i.e., $V^* \asymp V^{**}$. The following diagram illustrates the situation.

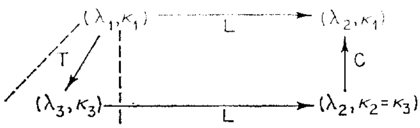

DIAGRAM 2.

We calculate the quantities which appear in this description. It follows from $A_{\lambda_3}^{\kappa_3} \leqslant \lambda_3^{\kappa_3}(V_1/\lambda_1^{\kappa_1})V$ by L that $A_{\lambda_2}^{\kappa_3} \leqslant V_2 = \lambda_2^{\kappa_3}(V_1/\lambda_1^{\kappa_1})\, V(\Lambda_3/\Lambda_2)^{\kappa_3}$, and then $V^* = \lambda_2^{\kappa_1}(V_1/\lambda_1^{\kappa_1})\, V(\Lambda_3/\Lambda_2)^{\kappa_3}$ (by C), whereas $V^{**} = \lambda_2^{\kappa_1}(V_1/\lambda_1^{\kappa_1})(\Lambda_1/\Lambda_2)^{\kappa_1}$ (by L). It follows from $V^* \asymp V^{**}$ that

$$\Lambda_2^{\kappa_1-\kappa_3} \asymp \Lambda_1^{\kappa_1} V^{-1} \Lambda_3^{-\kappa_3}, \tag{4}$$

and it follows from (4) and the expression for V_2 that

$$\frac{V_2}{\lambda_2^{\kappa_3}} \Lambda_2^{\kappa_1} \asymp \frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}. \tag{5}$$

Theorem T can now be formulated as follows.

Given two Riesz-means (λ_1, κ_1), (λ_3, κ_3), $\Lambda_1 \leqslant \Lambda_3$, $\kappa_3 < \kappa_1$, and given V with $1 \leqslant V \leqslant \Lambda_1^{\kappa_1}/\Lambda_3^{\kappa_3}$, suppose that λ_2 and V_2 satisfy (4) and (5). Then $A_{\lambda_1}^{\kappa_1} \leqslant V_1$, $A_{\lambda_2}^{\kappa_3} \leqslant V_2$ imply $A_{\lambda_3}^{\kappa_3} \leqslant \lambda_3^{\kappa_3}(V_1/\lambda_1^{\kappa_1})V$. We will show that a suitable combination of C, L (LC) and T always leads from (1) to the "minimal" estimate (2) (under the restrictions on λ_i, V_i and κ_i which we mentioned earlier). Here, the precise meaning of "minimal" is the following: V_3 will be called a minimal bound for $A_{\lambda_3}^{\kappa_3}$ if (2) holds, and if also $V_3 \leqslant U$ for every U of the property, that (1) implies $A_{\lambda_3}^{\kappa_3} \leqslant U$.

We are going to discuss now the relations between Theorems C, L, T and known results. The First and Second Theorem of Consistency (see e.g., [1, 5, 6, 8]), The Limitation Theorem (see, e.g., [1, Theorem 1.61], [5, Theorems 21, 22]), The Convexity Theorem of M. Riesz (see, e.g., [1, Theorem 1.71; 9; 10]), a theorem of Chandrasekharan and Minakshisundaram, denoted by C–M ([1, Theorem 2.41], it generalizes earlier results by Zygmund [11]) and a theorem by Zygmund, which is, in extended form, Theorem 2.61 of [1].

For $\kappa_3 \geqslant \kappa_1$, Theorem C is a combination of the first and second theorem of consistency, and for $\kappa_3 < \kappa_1$ it follows from C–M. Theorem L is, for $\lambda_3 = \lambda_1$, the Limitation Theorem, and for $\Lambda_3 \geqslant \Lambda_1$, it follows from C–M. (The connections between Theorems C, L and Theorem C–M will be shown in our later discussion of the Theorem C–M.) Theorem LC generalizes Theorem 2.61 of [1].

Theorem T is new, but some of its consequences are known: The Convexity Theorem is a combination of Theorems LC (or L, $\kappa_3 = \kappa_1$) and T. Its structure is: For $0 \leqslant \kappa_2 < \kappa_3 < \kappa_1$,

$$A_\lambda^{\kappa_1} \preccurlyeq V_1, \ A_\lambda^{\kappa_2} \preccurlyeq V_2 \quad \text{imply} \quad A_\lambda^{\kappa_3} \preccurlyeq V_3 = V_1^{(\kappa_3-\kappa_2)/(\kappa_1-\kappa_2)} V_2^{(\kappa_1-\kappa_3)/(\kappa_1-\kappa_2)},$$

and we may assume that $V_1/\lambda^{\kappa_1} \preccurlyeq V_2/\lambda^{\kappa_2} \preccurlyeq (V_1/\lambda^{\kappa_1})\, \Lambda^{\kappa_1-\kappa_2}$ (otherwise the theorem is of Abelian nature and follows from C or L).

Let[8] $\lambda_1 = \lambda$, $\lambda_3 = \lambda$, $V = (\lambda^{\kappa_1-\kappa_2} V_2/V_1)^{(\kappa_1-\kappa_3)/(\kappa_1-\kappa_2)}$, $\Lambda_2 \asymp \Lambda V^{1/(\kappa_3-\kappa_1)}$. It follows from Theorem LC that $A_{\lambda_2}^{\kappa_3} \preccurlyeq V_2{}^* = \lambda_2^{\kappa_3}(V_2/\lambda^{\kappa_2})(\Lambda/\Lambda_2)^{\kappa_2}$; the assumptions of Theorem T (with $V_2{}^*$ in place of V_2) are now satisfied, and it follows from this theorem that $A_\lambda^{\kappa_3} = A_{\lambda_3}^{\kappa_3} \preccurlyeq \lambda^{\kappa_3}(V_1/\lambda^{\kappa_1})V = V_3$, i.e., the Convexity Theorem follows.

The following diagram illustrates this proof:

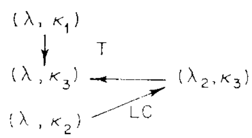

DIAGRAM 3.

According to the diagram we understand T as a stronger form of the Convexity Theorem, where the (λ, κ_2)-hypothesis is replaced by the weaker (λ_2, κ_3)-hypothesis which is even necessary for the conclusion.

Theorem C–M is of the following structure:

[8] With regard to the existence of Λ_2 we note the following: If $0 < f \in L$, $\int^\infty dt/f(t) = \infty$, then there is a λ satisfying (3) such that $\Lambda \sim f$. In fact, there is $F \in L$ such that $F \sim \int^x dt/f(t)$ (see [3]), and $\lambda = e^F$ satisfies $\Lambda \sim f$ (see [2, Theorem 21]).

Suppose that $\kappa_3 < \kappa_1$, $\Lambda_1 \leqslant \Lambda_3$, then

$$A^{\kappa_1}_{\lambda_1} \leqslant V_1, \; A^{\kappa_3}_{\lambda_1} \leqslant V_2 \quad \text{imply} \quad A^{\kappa_3}_{\lambda_3} \leqslant V_3 = \lambda_3^{\kappa_3}\left(\frac{V_1}{\lambda_1^{\kappa_1}} + \frac{V_2}{\lambda_1^{\kappa_3}}\left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_3}\right).$$

The logical structure of this theorem and its proof is indicated by the following diagram:

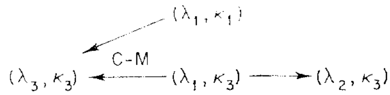

DIAGRAM 4.

$A^{\kappa_1}_{\lambda_1} \leqslant V_1$ implies $A^{\kappa_3}_{\lambda_1} \leqslant V_2 = \lambda_1^{\kappa_3}(V_1/\lambda_1^{\kappa_1})\,\Lambda_1^{\kappa_1-\kappa_3}$ (by Theorem L with $\lambda_3 = \lambda_1$); therefore, as was mentioned before, Theorems C and L (if $\kappa_3 < \kappa_1$, $\Lambda_1 \leqslant \Lambda_3$) are consequences of Theorem C–M.

In the discussion of the "Tauberian contents" of Theorem C–M we may assume that $\Lambda_3^{\kappa_3} \leqslant \Lambda_1^{\kappa_1}$ and also, that both terms in V_3 are of equal order (increase V_1 or V_2 if necessary), i.e., we may assume that $V_1/\lambda_1^{\kappa_1} \asymp (V_2/\lambda_1^{\kappa_3})(\Lambda_1/\Lambda_3)^{\kappa_3}$. We now introduce λ_2 through $\Lambda_2^{\kappa_1-\kappa_3} \asymp \Lambda_1^{\kappa_1}\Lambda_3^{-\kappa_3}$; then $A^{\kappa_3}_{\lambda_1} \leqslant V_2$ and Theorem L (or LC) imply $A^{\kappa_3}_{\lambda_2} \leqslant \lambda_2^{\kappa_3}(V_2/\lambda_1^{\kappa_3})(\Lambda_1/\Lambda_2)^{\kappa_3} = V_2{}^*$, and Theorem T (with $V = 1$, $V_2{}^*$ in place of V_2) shows that $A^{\kappa_3}_{\lambda_3} \leqslant V_3$, i.e., this part of Theorem C–M is a consequence of Theorems L and T. Accordingly, we may view T as a stronger form of the essential case of Theorem C–M, where the (λ_1, κ_3)-hypothesis is replaced by the weaker (λ_2, κ_3)-hypothesis. Observe that both of these conditions are necessary for the conclusion and that the (λ_2, κ_3)-hypothesis is the weakest condition of this kind.

In Section 1 of this paper we will give some auxiliary results on L-functions. Section 2 is devoted to the proof of the Abelian and Tauberian theorems. It turns out that we need three Abelian Theorems, denoted by A_1, A_2, A_3, whose logical structure is indicated by the following diagram:

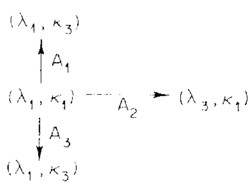

DIAGRAM 5.

All other Abelian Theorems follow from these special ones in combination with the Tauberian Theorem T. The key to Theorems A_1, A_2, A_3 and T are Theorem 1 (the sharpened Riesz mean-value theorem) and especially

Theorem 2 (which describes the influence of V_1 on parts of $A_{\lambda_3}^{\kappa_3}$). In Section 3 we prove Theorems C, L, LC. Combinations of these theorems with Theorem T (similarly to the preceding discussion of the Convexity Theorem) lead to Theorems 3 and 4, which form the basis of the main Theorem 5 (Section 4). This theorem solves the problem which was laid out at the beginning of this introduction. For a complete proof of Theorem 5 we must construct counterexamples which show that the estimates V_3 of Theorem 5 are minimal bounds. These counterexamples are also given in Section 4. We assume in Theorem 5 that the functions V_1, V_2 do not increase or decrease too fast. The concluding Section 5 indicates how Theorem 5 changes when V_1, V_2 increase or decrease more rapidly.

We conclude this introduction with a comment on the "o-theorems" or "mixed" theorems of Footnote 2. If, for instance, $A_{\lambda_1}^{\kappa_1} \preccurlyeq V_1$ in (1) is replaced by $A_{\lambda_1}^{\kappa_1} \prec V_1$ it seems natural to reduce this new case to the former by writing $A_{\lambda_1}^{\kappa_1}(x) \preccurlyeq \epsilon(x)\, V_1(x)$, $\epsilon(x) \to 0$, i.e., by replacing V_1 by ϵV_1 in (1). Unfortunately, the class L does not contain functions which decrease very slowly (see [2, 4.44]), so that this approach to "o-theorems" is ruled out. Instead, we will use the fact that $A_{\lambda_1}^{\kappa_1} \prec V_1$ implies $|A_{\lambda_1}^{\kappa_1}(x)| \leqslant \epsilon V_1(x)$, $x \geqslant x_0(\epsilon)$ for every constant $\epsilon > 0$, and we will show that this constant ϵ (or a function of it) will also appear in the corresponding V_3. Obviously, in doing so we must control the constants which appear in V_3, in other words, we must prove that our estimates V_3 are uniform in a certain sense. This remark explains why we formulate some of the following lemmas in Section 1 with numerical constants.

1. Auxiliary Results on L-Functions

The following lemmas contain statements on functions λ, λ_3, and we assume throughout that λ_3 satisfies (3). By $\bar{\lambda}_3$ we will denote the inverse function of λ_3, and we will write $\bar{x} = \bar{\lambda}_3(\frac{1}{2}\lambda_3(x))$.

If functions $f_1(x), f_2(x)$ are defined for all large x, we will write $f_1(x) \dot{\leqslant} f_2(x)$ if $f_1(x) \leqslant f_2(x)$, $x \geqslant x_0$, holds for some $x_0 > 0$ (and similarly $\dot{<}$, $\dot{\geqslant}$, $\dot{>}$, $\dot{=}$).

Lemma 1. *Suppose that λ satisfies* (3), *and that $\Lambda \dot{\leqslant} \Lambda_3$. Then*

$$|(d/dt)(\lambda_3'/\lambda')| \dot{\leqslant} 3(\lambda_3'(t)/\lambda(t)). \tag{6}$$

Proof. We have $\Lambda\lambda_3' \dot{\leqslant} \Lambda_3\lambda_3' = \lambda_3$, and it follows (compare [2, Theorem 21]) that $|(\Lambda\lambda_3')'| \dot{\leqslant} 2\lambda_3'$, which proves (6) since $(\lambda_3'/\lambda')' = (\Lambda\lambda_3'/\lambda)' = (\Lambda\lambda_3')'/\lambda - \lambda_3'/\lambda$.

LEMMA 2. *Suppose that* $0 \dot{<} \lambda \in L$, *and that*

$$0 \leqslant \lambda_3(x) - \lambda_3(t) \leqslant \lambda_3(x) \min\left(\frac{1}{2}, \frac{|\Lambda(x)|}{\Lambda_3(x)}\right) = \lambda_3(x) f(x). \tag{7}$$

Then,

$$e^{-4} \dot{\leqslant} \lambda(t)/\lambda(x) \dot{\leqslant} e^4. \tag{8}$$

If, in addition, $\lambda \to \infty$, *then*

$$e^{-4} \dot{\leqslant} \Lambda(t)\, \lambda_3'(t)/\Lambda(x)\, \lambda_3'(x) \dot{\leqslant} e^4, \tag{9}$$

$$e^{-8} \dot{\leqslant} \frac{\lambda'(t)}{\lambda_3'(t)} \Big/ \frac{\lambda'(x)}{\lambda_3'(x)} \dot{\leqslant} e^8. \tag{10}$$

Proof. We first prove (8) (cf. also [2, Theorem 31]). Suppose that $\lambda \uparrow$ [9]. Applying the mean-value theorem we find that

$$A = \log \lambda(x)/\lambda(t) = \log \lambda(\bar{\lambda}_3(\lambda_3(x)))/\lambda(\bar{\lambda}_3(\lambda_3(t))) = \frac{\lambda_3(x) - \lambda_3(t)}{\Lambda(\xi)\, \lambda_3'(\xi)}$$

for some ξ satisfying $\bar{x} \leqslant t \leqslant \xi \leqslant x$.

If $\Lambda(x)/\Lambda_3(x) \to \alpha > 0$, $\alpha \leqslant \infty$, then

$$A \leqslant \frac{\lambda_3(x) f(x)}{\Lambda(\xi)\, \lambda_3'(\xi)} \dot{\leqslant} 2\, \frac{\lambda_3(x) \min(\frac{1}{2}, \Lambda(\xi)/\Lambda_3(\xi))}{\Lambda(\xi)\, \lambda_3'(\xi)} \leqslant 2\, \frac{\lambda_3(x)}{\lambda_3(\xi)} \leqslant 2\, \frac{\lambda_3(x)}{\lambda_3(\bar{x})} = 4.$$

If $\Lambda(x)/\Lambda_3(x) \to 0$ (hence $\downarrow$ for large x), then

$$\Lambda(\xi)\, \lambda_3'(\xi) = \frac{\Lambda(\xi)}{\Lambda_3(\xi)}\, \lambda_3(\xi) \geqslant \frac{\Lambda(\xi)}{\Lambda_3(\xi)}\, \lambda_3(t) \dot{\geqslant} \frac{\Lambda(x)}{\Lambda_3(x)}\, \frac{\lambda_3(x)}{2},$$

hence

$$A \dot{\leqslant} \frac{\lambda_3(x)(\Lambda(x)/\Lambda_3(x))}{\Lambda(\xi)\, \lambda_3'(\xi)} \leqslant 2.$$

This proves (8) in this case. If $\lambda \downarrow$, then $\tilde{\lambda} = 1/\lambda \uparrow$, and $\tilde{\Lambda} = |\Lambda|$, i.e., this case follows from the case $\lambda \uparrow$.

In order to obtain (9) we apply (8) to the function $\lambda^* = \Lambda\lambda_3'$, and (9) follows if we show that $\min(\frac{1}{2}, \Lambda/\Lambda_3) \dot{\leqslant} |\Lambda^*|/\Lambda_3$. If $\lambda^* \uparrow$, then the assumption $(\Lambda^*/\Lambda_3) \dot{\leqslant} \frac{1}{2}$ would imply $\lambda^* \dot{\geqslant} c\lambda_3^2$ $(c > 0)$, and in turn $\lambda \leqslant 1$; hence $\Lambda^*/\Lambda_3 \geqslant \frac{1}{2}$. If $\lambda^* \downarrow$, then $a(x) = \lambda^*(\bar{\lambda}_3(x)) \downarrow$; therefore, $a'(\lambda_3(x)) = \lambda^{*\prime}(x)/\lambda_3'(x) \uparrow 0$, and then $\Lambda \dot{\leqslant} |\Lambda(\lambda_3'/\lambda^{*\prime})| = |\Lambda^*|$.

Inequality (10) follows from (8) and (9) because $\lambda'/\lambda_3' = \lambda/\lambda^*$.

[9] $\uparrow(\downarrow)$ denotes ultimately increasing (decreasing) in the wider sense.

Remark. This proof also shows that (8), (9) and (10) remain true (possibly with new constants) when $|\Delta|/\Delta_3$ in (7) is replaced by $c(|\Delta|/\Delta_3)$, $c > 0$.

LEMMA 3. *Suppose that* $0 \dot{<} \lambda \in L$, *and that* $\lambda \succ \lambda_3^{-\Delta}$ (*resp.* $\lambda \prec \lambda_3^{\Delta}$) *for some* $\Delta > 0$. *Then there exists* $K > 0$ *such that*

$$\lambda(t)/\lambda(x) \dot{\leqslant} K \quad (\textit{resp. } \lambda(t)/\lambda(x) \dot{\geqslant} K) \qquad \textit{if} \quad \bar{x} \leqslant t \leqslant x. \tag{11}$$

Proof. We have $\lambda\lambda_3^{\Delta} \uparrow$ (resp. $\lambda\lambda_3^{-\Delta} \downarrow$).

LEMMA 4. *Suppose that* $0 \dot{<} \lambda \in L$. *Then*

$$\int^x \lambda(t)\,\lambda_3'(t)\,dt \begin{Bmatrix} \succ \\ \succcurlyeq \\ \preccurlyeq \\ \prec \end{Bmatrix} \lambda(x)\,\lambda_3(x), \qquad \textit{if} \quad \begin{cases} \lambda \prec \lambda_3^{\delta-1}, & \textit{for every} \quad \delta > 0, \\ \lambda \prec \lambda_3^{\Delta}, & \textit{for some} \quad \Delta > 0, \\ \lambda \succ \lambda_3^{\delta-1}, & \textit{for some} \quad \delta > 0, \\ \lambda \succ \lambda_3^{\Delta}, & \textit{for every} \quad \Delta > 0. \end{cases}$$

Proof. The statements on $\prec$, $\succ$ follow from [3] (note that $\int^x \lambda\lambda_3'\,dt = \int^{\lambda_3(x)} \lambda(\bar{\lambda}_3(v))\,dv$) or from [2, Theorem 25], and the remaining statements follow from Lemma 3 ($\succcurlyeq$) and from $\lambda\lambda_3^{1-\delta} \uparrow$ ($\preccurlyeq$).

LEMMA 5. *Suppose that* $0 \dot{<} \lambda \in C[0, \infty)$, *that* $\lambda \in L$, *and that* $\kappa > 0$. *Then*

$$\int_0^x (\lambda_3(x) - \lambda_3(t))^{\kappa-1}\,\lambda_3'(t)\,\lambda(t)\,dt \asymp \lambda_3^{\kappa-1}(x) \int_0^x \lambda_3'\lambda\,dt \succcurlyeq \lambda_3^{\kappa}(x)\,\lambda(x),$$
$$\textit{if} \quad \lambda \preccurlyeq \lambda_3^{\Delta} \qquad \textit{for some} \quad \Delta > 0, \tag{12}$$
$$\int_0^x (\lambda_3(x) - \lambda_3(t))^{\kappa-1}\,\lambda_3'(t)\,\lambda(t)\,dt \dot{\leqslant} C(\kappa, \delta)\,\lambda_3^{\kappa}(x)\,\lambda(x),$$
$$\textit{if} \quad \lambda \succ \lambda_3^{\delta-1} \qquad \textit{for some} \quad \delta > 0. \tag{13}$$

Proof. Formula (12) can be proven in the following way: If $\lambda \preccurlyeq \lambda_3^{-2}$, then (12) is obvious. If $\lambda_3^{-2} \preccurlyeq \lambda \preccurlyeq \lambda_3^{\Delta}$, then it follows from Lemma 3 that $\lambda(t) \asymp \lambda(x)$ if $\bar{x} \leqslant t \leqslant x$, and we have

$$\begin{aligned} &\int_0^x (\lambda_3(x) - \lambda_3(t))^{\kappa-1}\,\lambda_3'(t)\,\lambda(t)\,dt \\ &\quad \asymp \lambda_3^{\kappa-1}(x) \int_0^{\bar{x}} \lambda_3'(t)\,\lambda(t)\,dt + \lambda(x) \int_{\bar{x}}^x (\lambda_3(x) - \lambda_3(t))^{\kappa-1}\,\lambda_3'(t)\,dt \\ &\quad \asymp \lambda_3^{\kappa-1}(x) \left(\int_0^{\bar{x}} \lambda_3'(t)\,\lambda(t)\,dt + \lambda(x)\,\lambda_3(x) \right) \\ &\quad \asymp \lambda_3^{\kappa-1}(x) \int_0^x \lambda_3'(t)\,\lambda(t)\,dt. \end{aligned}$$

The inequality in (12) follows from Lemma 4. In order to prove (13) we may proceed on similar lines if we observe that the constants in Lemmas 3 and 4 depend on Δ, δ only. More directly the result follows from

$$\int_{x_0}^{x} (\lambda_3(x) - \lambda_3(t))^{\kappa-1} \lambda_3'(t)\, \lambda(t)\, dt \leqslant \lambda(x)\, \lambda_3^{1-\delta}(x) \int_{x_0}^{x} (\lambda_3(x) - \lambda_3(t))^{\kappa-1} \lambda_3'(t)\, \lambda_3^{\delta-1}(t)\, dt.$$

LEMMA 6. *Suppose that λ satisfies* (3), *and that $\kappa > 0$. Then*

$$\int_{y}^{x} (\lambda(x) - \lambda(t))^{\kappa-1} \lambda'(t)\, dt \asymp \min\left(\lambda^{\kappa}(x), \left((\lambda_3(x) - \lambda_3(y)) \frac{\lambda'(x)}{\lambda_3'(x)}\right)^{\kappa}\right) \tag{14}$$

as $x \to \infty$, $\bar{x} \leqslant y \leqslant x$.

Proof. The integral is $(1/\kappa)(\lambda(x) - \lambda(y))^{\kappa}$. If

$$\lambda_3(x) - \lambda_3(y) \leqslant \lambda_3(x)(\Lambda(x)/\Lambda_3(x)),$$

then for suitable $\xi \in [y, x]$

$$\begin{aligned}\lambda(x) - \lambda(y) &= \lambda(\bar{\lambda}_3(\lambda_3(x))) - \lambda(\bar{\lambda}_3(\lambda_3(y)))\\ &= (\lambda_3(x) - \lambda_3(y)) \frac{\lambda'(\xi)}{\lambda_3'(\xi)} \asymp (\lambda_3(x) - \lambda_3(y)) \frac{\lambda'(x)}{\lambda_3'(x)},\end{aligned}$$

by (10) (note that $\lambda_3(x) - \lambda_3(y) \leqslant \frac{1}{2}\lambda_3(x)$). If

$$\lambda_3(x) - \lambda_3(y) \geqslant \lambda_3(x)(\Lambda(x)/\Lambda_3(x)),$$

then

$$\frac{\lambda_3(x)}{2} = \lambda_3(x) - \lambda_3(\bar{x}) \geqslant \lambda_3(x) - \lambda_3(y) \geqslant \lambda_3(x) \frac{\Lambda(x)}{\Lambda_3(x)};$$

hence, introducing $x^* = \bar{\lambda}_3(\lambda_3(x) - \lambda_3'(x)\, \Lambda(x)) \geqslant y \geqslant \bar{x}$, $\lambda(x) - \lambda(x^*) = \lambda_3'(x)\, \Lambda(x)\, \lambda'(\xi)/\lambda_3'(\xi)$ with $\xi \in [x^*, x]$. Therefore, by use of (10)

$$\lambda(x) \geqslant \lambda(x) - \lambda(y) \geqslant \lambda(x) - \lambda(x^*) \asymp \lambda_3'(x)\, \Lambda(x)\, \lambda'(\xi)/\lambda_3'(\xi) \geqslant \lambda(x).$$

Thus, in this case,

$$(\lambda(x) - \lambda(y))^{\kappa} \asymp \lambda^{\kappa}(x).$$

LEMMA 7. *Suppose that λ satisfies* (3), *and that $\lambda'(x)/\lambda_3'(x)$ is monotone for $x \geqslant x_0$. Then*

$$\frac{\lambda(x) - \lambda(t)}{\lambda_3(x) - \lambda_3(t)} \uparrow (\downarrow) \quad \textit{in} \quad t \in [x_0, x] \qquad \textit{if} \quad \frac{\lambda'(x)}{\lambda_3'(x)} \uparrow (\downarrow). \tag{15}$$

If $\lambda'(x)/\lambda_3'(x)\uparrow$, then for every $\alpha < 1$,

$$\frac{\lambda_3'(t)}{\lambda'(t)}\left(\frac{\lambda(x)-\lambda(t)}{\lambda_3(x)-\lambda_3(t)}\right)^{\alpha}\downarrow \quad in \quad t\in[x_1, x], \qquad x_1 = x_1(\alpha). \tag{16}$$

Proof. Writing $y = \lambda_3(x)$, $\tau = \lambda_3(t)$, $\mu(\tau) = \lambda(\bar{\lambda}_3(\tau))$, we have

$$\mu'(\tau) = \lambda'(\bar{\lambda}_3(\tau))/\lambda_3'(\bar{\lambda}_3(\tau)), \qquad \frac{\lambda(x)-\lambda(t)}{\lambda_3(x)-\lambda_3(t)} = \frac{\mu(y)-\mu(\tau)}{y-\tau}$$

(and μ and its derivatives are L-functions of the variable $\bar{\lambda}_3(\tau)$). Statement (15) follows immediately from

$$\frac{\mu(y)-\mu(\tau)}{y-\tau} = \int_0^1 \mu'(\tau + w(y-\tau))\,dw = \int_0^1 \mu'(wy + \tau(1-w))\,dw$$

and from the monotonicity of μ'. In proving (16) we may assume that $\alpha\in(0, 1)$ (if $\alpha \leqslant 0$, then (16) follows from (15)), and (16) is true if

$$A(y,\tau) = \frac{1}{(\mu'(\tau))^{\beta}}\frac{\mu(y)-\mu(\tau)}{y-\tau} = \frac{1}{(\mu'(\tau))^{\beta}}\int_0^1 \mu'(\tau + w(y-\tau))\,dw\downarrow,$$

for every fixed $\beta > 1$ and for $y(\beta) < \tau\uparrow y$. Writing $g(\tau) = \mu''(\tau)/\mu'(\tau)$ we have

$$\begin{aligned} A_\tau &= \frac{\partial}{\partial\tau}A(y,\tau) \\ &= A(y,\tau)\left(\frac{\int_0^1 g(\tau + w(y-\tau))\,\mu'(\tau + w(y-\tau))(1-w)\,dw}{\int_0^1 \mu'(\tau + w(y-\tau))\,dw} - \beta g(\tau)\right). \end{aligned}$$

Integrating by parts we find

$$\begin{aligned} &\int_0^1 g(\tau + w(y-\tau))\,\mu'(\tau + w(y-\tau))(1-w)\,dw \\ &\qquad = -\frac{\mu'(\tau)}{y-\tau} + \frac{1}{y-\tau}\int_0^1 \mu'(\tau + w(y-\tau))\,dw, \end{aligned}$$

and $A_\tau \leqslant 0$ if $g(\tau) \geqslant 1/(y-\tau)$. Therefore, we must only discuss the case $g(\tau) \leqslant 1/(y-\tau)$, and we distinguish between $g\downarrow$ and $g\uparrow$. In the first case $A_\tau \leqslant 0$ because

$$\int_0^1 g\mu'(1-w)\,dw \leqslant g(\tau)\int_0^1 \mu'(\tau + w(y-\tau))\,dw,$$

and the case $g \uparrow$, $g(\tau) \leqslant 1/(y-\tau)$ remains. In this case $(1/g)' \to 0$, and in particular $|(1/g)'| \leqslant \delta = 1 - 1/\beta$ for all large τ (the bound depends on β). Then

$$\frac{1}{g(\tau)} - \frac{1}{g(y)} = -\int_\tau^y \left(\frac{1}{g}\right)' d\tau \leqslant \delta(y-\tau) \leqslant \frac{\delta}{g(\tau)},$$

hence $g(y) \leqslant \beta g(\tau)$, and $A_\tau \leqslant 0$ follows from

$$\int_0^1 g\mu'(1-w)\,dw \leqslant g(y)\int_0^1 \mu'(\tau + w(y-\tau))\,dw.$$

2. Abelian and Tauberian Theorems

Throughout the paper the index κ of Riesz means is in [0, 1]. Suppose that $\kappa > 0$, $0 \leqslant \xi \leqslant x$, that λ satisfies (3), and that $A \in M$. Then we define

$$A_\lambda{}^\kappa(x, \xi) = \int_0^\xi (\lambda(x) - \lambda(t))^{\kappa-1}\,\lambda'(t)\,A(t)\,dt.$$

In what follows, V, V_1, V_2 will denote functions which are nonnegative and belong to $C[0, \infty)$ and L. We introduce the condition

$$V_i\lambda_i^{1-\kappa_i} \succ 1, \qquad V_i\lambda_i^{1-\epsilon_i} \succ 1 \qquad \text{for some} \quad \epsilon_i \in (0,1)\ ^{10} \qquad (17_i)$$

which will be of central importance.

Our Abelian theorems will lead from assumptions $|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1$ to conclusions $|A_{\lambda_3}^{\kappa_3}| \dot{\leqslant} c_1V_3$. If V_3 satisfies (17_3), then it follows from

$$|A_{\lambda_3}^{\kappa_3}(x, x_0)| \leqslant \operatorname*{ess\,sup}_{0\leqslant t\leqslant x_0} |A(t)|\,(\lambda_3(x) - \lambda_3(x_0))^{\kappa_3-1}\,\lambda_3(x_0), \qquad (18)$$

that[11]

$$|A_{\lambda_3}^{\kappa_3}(x, x_0)/V_3(x)| \to 0 \qquad \text{as} \qquad x \to \infty, \qquad (19)$$

hence, in order to prove an Abelian theorem of this type we need only show that $|A_{\lambda_3}^{\kappa_3}(x) - A_{\lambda_3}^{\kappa_3}(x, x_0)| \dot{\leqslant} c_2V_3(x)$, where $0 < c_2 < c_1$.

[10] For $\kappa_i > 0$ the second condition follows from the first.

[11] It is obvious, that (17_3) with $\succcurlyeq$ in place of $\succ$ would be sufficient as long as we are concerned with "O-theorems." Condition (17_i) in its present form is required to obtain also "o-theorems."

THEOREM 1 (*Riesz mean-value theorem with normalizing factor*). *Suppose that* $0 < \kappa_1 \leqslant 1$, $A \in M$. *Then*

$$A_{\lambda_1}^{\kappa_1}(x, \xi) = \left(\frac{\lambda_1(\xi')}{\lambda_1(x)}\right)^{1-\kappa_1} A_{\lambda_1}^{\kappa_1}(\xi') \qquad \textit{for some} \quad \xi' \in [0, \xi]. \tag{20}$$

For a proof see, e.g., [7].

The following statement is a consequence of (20) (discuss the cases ξ' near 0 and ξ' large separately): If V_1 satisfies (17_1), then

$$|A_{\lambda_1}^{\kappa_1}(x)| \dot{\leqslant} V_1(x) \quad \text{implies} \quad |A_{\lambda_1}^{\kappa_1}(x, \xi)| \leqslant \left(\frac{\lambda_1(\eta)}{\lambda_1(x)}\right)^{1-\kappa_1} V_1(\eta), \tag{21}$$

whenever $\xi \leqslant \eta \leqslant x$, $\eta \geqslant x_0$ (x_0 independent of ξ and x).

THEOREM A_1 (*First Theorem of Consistency*). *Suppose that* (17_1) *holds, and that*

$$0 \leqslant \kappa_1 < \kappa_3 \leqslant 1, \qquad A \in M.$$

Then

$$|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1 \qquad \textit{implies} \qquad |A_{\lambda_1}^{\kappa_3}| \dot{\leqslant} KV_3,$$

$$V_3 = \lambda_1^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \quad \textit{where} \quad K = \begin{cases} 1 & \textit{if } \kappa_1 > 0, \\ \Gamma(\kappa_3)\,\Gamma(\epsilon_1)/\Gamma(\kappa_3 + \epsilon_1) & \textit{if } \kappa_1 = 0. \end{cases} \tag{22}$$

Proof.[12] If $\kappa_1 > 0$, then (from the mean-value theorem for integrals)

$$A_{\lambda_1}^{\kappa_3}(x) = \lambda_1^{\kappa_3-\kappa_1}(x)\, A_{\lambda_1}^{\kappa_1}(x, \xi) \tag{23}$$

and (22) follows from (21) (for $\eta = x$).

If $\kappa_1 = 0$, then

$$|A_{\lambda_1}^{\kappa_3}(x)| \leqslant \operatorname*{ess\,sup}_{0 \leqslant t \leqslant x} |A(t)\, \lambda_1^{1-\epsilon_1}(t)| \int_0^x (\lambda_1(x) - \lambda_1(t))^{\kappa_3-1}\, \lambda_1'(t)\, \lambda_1^{\epsilon_1-1}(t)\, dt$$

$$= \frac{\Gamma(\epsilon_1)\,\Gamma(\kappa_3)}{\Gamma(\epsilon_1 + \kappa_3)} \operatorname*{ess\,sup}_{0 \leqslant t \leqslant x} |A(t)\, \lambda_1^{1-\epsilon_1}(t)|\, (\lambda_1(x))^{\kappa_3+\epsilon_1-1}$$

and (22) follows from (17_1).

Two arguments will repeatedly be used in the following proofs, and we will discuss them beforehand.

[12] This theorem and its proof is a slight extension of well-known results; we indicate the proof to explain, e.g., the value of κ.

Suppose that $\lambda_j (j = 1, 2, 3)$ satisfy (3). Let

$$f_i(x) = \lambda_3(x) \min\left(\frac{1}{2}, \frac{\Lambda_i(x)}{\Lambda_3(x)}\right), \qquad x_i^* = \bar{\lambda}_3(\lambda_3(x) - f_i(x)) \quad (i = 1, 2).$$

Then it follows from Lemma 2 that $\varphi_i(x) = \lambda_i'(x)/\lambda_3'(x)$ satisfies

$$e^{-16} < \varphi_i(\alpha)/\varphi_i(\beta) < e^{16} \quad \text{whenever}^{13} \quad x_i^* \leqslant \alpha \leqslant \beta \leqslant x,\ x \text{ large.} \tag{24}$$

Suppose $0 \leqslant x_1 \leqslant x_2 \leqslant x$, and consider the integral

$$I = \int_{x_1}^{x_2} (\lambda(x) - \lambda(t))^{\kappa-1} \lambda'(t) A(t) a(t) b_1(t) \cdots b_p(t)\, dt, \qquad 0 < \kappa \leqslant 1,$$

where λ satisfies (3), $A \in M$, $0 \leqslant a \uparrow$, $b_1 \cdots b_p$ monotone and nonnegative. Then a repeated application of the mean-value theorem for integrals shows that

$$I = a(x_2)\, b_1(\xi_1) \cdots b_p(\xi_p) \int_\rho^\sigma (\lambda(x) - \lambda(t))^{\kappa-1} \lambda'(t) A(t)\, dt,$$

$$\xi_1, ..., \xi_p, \rho, \sigma \in [x_1, x_2]$$

and this implies

$$|I| \leqslant 2a(x_2)\, b_1(\xi_1) \cdots b_p(\xi_p) \sup_{0 \leqslant \xi \leqslant x} |A_\lambda^\kappa(x, \xi)|. \tag{25}$$

THEOREM A_2. *Suppose that* (17_1) *holds, and that*

$$0 \leqslant \kappa_1 \leqslant 1, \qquad A \in M, \qquad \Lambda_3 \dot{\leqslant} \Lambda_1 .$$

Then[14]

$$|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1 \quad \textit{implies} \quad |A_{\lambda_3}^{\kappa_1}| \dot{\leqslant} 5V_3, \qquad V_3 = \lambda_3^{\kappa_1} \frac{V_1}{\lambda_1^{\kappa_1}} \left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_1}. \tag{26}$$

Proof. We may assume that $\kappa_1 > 0$. The inequality $\Lambda_3 \dot{\leqslant} \Lambda_1$ implies $\lambda_3 \dot{\geqslant} c\lambda_1$ for some $c > 0$, and it follows that (17_1) implies (17_3) (with $\kappa_3 = \kappa_1$), and that $\lambda_3'/\lambda_1' = (\Lambda_1/\Lambda_3)(\lambda_3/\lambda_1) \dot{\geqslant} c$. Let,

$$\begin{aligned} I &= A_{\lambda_3}^{\kappa_1}(x) - A_{\lambda_3}^{\kappa_1}(x, x_0) \\ &= \int_{x_0}^{x} (\lambda_1(x) - \lambda_1(t))^{\kappa_1 - 1} \lambda_1'(t) A(t) \left(\frac{\lambda_3(x) - \lambda_3(t)}{\lambda_1(x) - \lambda_1(t)}\right)^{\kappa_1 - 1} \frac{dt}{\varphi_1(t)}, \end{aligned}$$

[13] $\varphi_i(\alpha)/\varphi_i(\beta) = (\varphi_i(\alpha)/\varphi_i(x))(\varphi_i(x)/\varphi_i(\beta))$.

[14] The special case $V_3 = \lambda_3^{\kappa_1}$ is due to Zygmund [11] and [1, Theorem 2.61].

(x_0 sufficiently large). If $\varphi_1 \downarrow$, then it follows from Lemma 7 that $J = ((\lambda_3(x) - \lambda_3(t))/(\lambda_1(x) - \lambda_1(t)))^{\kappa_1 - 1}(1/\varphi_1(t)) \uparrow$ in t, and (25) shows that

$$|I| \leqslant 2\left(\frac{\lambda_3'(x)}{\lambda_1'(x)}\right)^{\kappa_1} \sup_{0 \leqslant \xi \leqslant x} |A_{\lambda_1}^{\kappa_1}(x, \xi)|.$$

If $\varphi_1 \uparrow$, then (15) and (25) (with $a = 1$) show that

$$|I| \dot{\leqslant} 2\left(\frac{\lambda_3'(\xi_1)}{\lambda_1'(\xi_1)}\right)^{\kappa_1 - 1} \frac{\lambda_3'(\xi_2)}{\lambda_1'(\xi_2)} \sup_{0 \leqslant \xi \leqslant x} |A_{\lambda_1}^{\kappa_1}(x, \xi)|, \qquad \xi_1, \xi_2 \in [x_0, x],$$

and we have $\lambda_3'(x)/\lambda_1'(x) \to d > 0$ in this case. In both cases we have (for x_0 sufficiently large)

$$|I| \dot{\leqslant} 4\left(\frac{\lambda_3'(x)}{\lambda_1'(x)}\right)^{\kappa_1} \sup_{0 \leqslant \xi \leqslant x} |A_{\lambda_1}^{\kappa_1}(x, \xi)|.$$

The statement (26) now follows from (21), $\eta = x$. (The factor 5 appears in (26) on account of (19).)

THEOREM A_3. *Suppose that* V_1 *satisfies* (17_1), *that*

$$0 \leqslant \kappa_3 < \kappa_1 \leqslant 1, \qquad A \in S,$$

and that[15]

$$V_1(x+1) \dot{\leqslant} cV_1(x), \qquad \Lambda_1 \dot{\geqslant} \alpha, \qquad \textit{for constants} \quad c > 0, \quad \alpha > 0.$$

Then

$$|A_{\lambda_1}^{\kappa_1}| \leqslant V_1 \quad \textit{implies} \quad |A_{\lambda_1}^{\kappa_3}| \dot{\leqslant} K_1 V_3, \qquad V_3 = \lambda_1^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1 - \kappa_3}, \tag{27}$$

for some K_1 *which depends on* c, κ_3 *and* α *only.*

This is essentially Theorem 1.61 of [1], and we omit its proof (which uses Lemma 2).

Our next theorem is the essential tool for the proof of the Tauberian Theorem T. It exhibits the magnitude of $A_{\lambda_3}^{\kappa_3}(x, y)$, as far as it is controlled by V_1 only, in a certain range of y near x, and it turns out that $y = x_1{}^*$ is a critical choice.

THEOREM 2. *Suppose that* (17_1) *holds, and that*

$$0 < \kappa_3 \leqslant \kappa_1 \leqslant 1, \qquad A \in M, \qquad \Lambda_1 \dot{\leqslant} \Lambda_3.$$

[15] Here A is a step function with steps at the integers, and $V_1(x+1) \leqslant cV_1(x)$ guarantees that V_1 does not increase too much between two integers.

Then there is a numerical constant $K_2 > 0$[16] *such that* $| A_{\lambda_1}^{\kappa_1} | \dot{\leqslant} V_1$ *implies*

$$| A_{\lambda_3}^{\kappa_3}(x, x_1^*)| \dot{\leqslant} K_2 V_3 ,$$
$$V_3 = \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_3} + \int_0^x (\lambda_3(x) - \lambda_3(t))^{\kappa_3 - 1} \lambda_3'(t) \frac{V_1(t)}{\lambda_1^{\kappa_1}(t)} dt \tag{28}$$

if V_3 *satisfies* (17_3).[17]

Remark. The following proof will also show that

$$| A_{\lambda_1}^{\kappa_1} | \dot{\leqslant} V_1 \quad \text{implies} \quad | A_{\lambda_3}^{\kappa_3}(x, \bar{x})| \dot{\leqslant} K_2 V_3 , \qquad V_3 = \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \quad \text{if } \Lambda_1 \doteq \Lambda_3 . \tag{29}$$

Proof of Theorem 2. Throughout this proof we will assume that x_0 and x are sufficiently large.

We split the integral $A_{\lambda_3}^{\kappa_3}(x, x_1^*) - A_{\lambda_3}^{\kappa_3}(x, x_0)$ into two terms:

$$I_1 = \int_{x_0}^{\bar{x}} (\cdots)\, dt, \qquad I_2 = \int_{\bar{x}}^{x_1^*} (\cdots)\, dt.$$

We have

$$I_1 = (\lambda_3(x) - \lambda_3(\bar{x}))^{\kappa_3 - 1} \int_{\xi}^{\bar{x}} \lambda_3'(t) A(t)\, dt = \lambda_3^{\kappa_3 - 1}(\bar{x}) \int_{\xi}^{\bar{x}} \lambda_3'(t) A(t)\, dt,$$
$$(x_0 \leqslant \xi \leqslant \bar{x}),$$

and, by partial integration,

$$\lambda_3^{1-\kappa_3}(\bar{x})\, I_1 = \frac{1}{\varphi_1(\bar{x})} \int_{\xi}^{\bar{x}} \lambda_1'(t) A(t)\, dt - \int_{\xi}^{\bar{x}} \left(\frac{1}{\varphi_1}\right)' dt \int_{\xi}^{t} \lambda_1'(\tau) A(\tau)\, d\tau.$$

It follows from (17_1) and (22) that

$$\left| \int_{\xi}^{t} \lambda_1'(\tau) A(\tau)\, d\tau \right| \leqslant 2\lambda_1^{1-\kappa_1}(t)\, V_1(t),$$

and we find from (6)

$$| I_1 | \leqslant 2\lambda_3^{\kappa_3 - 1}(\bar{x}) \left(\lambda_3(\bar{x}) \frac{\Lambda_1(\bar{x})}{\Lambda_3(\bar{x})} \frac{V_1(\bar{x})}{\lambda_1^{\kappa_1}(\bar{x})} + 3 \int_{\xi}^{\bar{x}} \lambda_3'(t) \frac{V_1(t)}{\lambda_1^{\kappa_1}(t)}\, dt\right).$$

[16] The proof will show that we may take $K_2 = 5e^{32}$.

[17] $V_3 \geqslant \lambda_3^{\kappa_3 - 1}(x) \int_0^x \lambda_3'(V_1/\lambda_1^{\kappa_1})\, dt$ shows that (17_3) holds if $\int^{\infty} \lambda_3'(V_1/\lambda_1^{\kappa_1})\, dt = \infty$.

The ultimate monotonicity of $V_1(t)/\lambda_1^{\kappa_1}(t)$ implies $\lambda_3(\bar{x})(V_1(\bar{x})/\lambda_1^{\kappa_1}(\bar{x})) \leqslant 2\int_0^x \lambda_3'(V_1/\lambda_1^{\kappa_1})\,dt$, and the estimate

$$|I_1| \leqslant 10\lambda_3^{\kappa_3-1}(\bar{x})\int_0^x \lambda_3' \frac{V_1}{\lambda_1^{\kappa_1}}\,dt \leqslant 20\int_0^x (\lambda_3(x)-\lambda_3(t))^{\kappa_3-1}\lambda_3'(t)\frac{V_1(t)}{\lambda_1^{\kappa_1}(t)}\,dt,$$

of I_1 follows. If $\Lambda_1 \doteq \Lambda_3$, then $(1/\varphi_1)' \doteq 0$, and (29) follows from the preceeding discussion of I_1.

Next, we have (by partial integration)

$$\begin{aligned}
I_2 &= \int_{\bar{x}}^{x_1^*} (\lambda_1(x)-\lambda_1(t))^{\kappa_1-1}\lambda_1'(t)\,A(t)(\lambda_1(x)-\lambda_1(t))^{1-\kappa_1}(\lambda_3(x)-\lambda_3(t))^{\kappa_3-1}\frac{dt}{\varphi_1(t)}\\
&= (\lambda_1(x)-\lambda_1(x_1^*))^{1-\kappa_1}\frac{f_1^{\kappa_3-1}(x)}{\varphi_1(x_1^*)}\int_{\bar{x}}^{x_1^*}(\lambda_1(x)-\lambda_1(t))^{\kappa_1-1}\lambda_1'(t)\,A(t)\,dt\\
&\quad -\int_{\bar{x}}^{x_1^*}\frac{d}{dt}\left\{(\lambda_1(x)-\lambda_1(t))^{1-\kappa_1}(\lambda_3(x)-\lambda_3(t))^{\kappa_3-1}\frac{1}{\varphi_1(t)}\right\}dt\\
&\quad \times\int_{\bar{x}}^{t}(\lambda_1(x)-\lambda_1(\tau))^{\kappa_1-1}\lambda_1'(\tau)\,A(\tau)\,d\tau.
\end{aligned}$$

It follows from $\lambda_1(x)-\lambda_1(x_1^*) = \lambda_1(\bar{\lambda}_3(\lambda_3(x))) - \lambda_1(\bar{\lambda}_3(\lambda_3(x_1^*)))$, that

$$(\lambda_1(x)-\lambda_1(x_1^*))^{1-\kappa_1} = (f_1(x)\,\varphi_1(\xi))^{1-\kappa_1}, \qquad x_1^* \leqslant \xi \leqslant x.$$

We have

$$f_1(x) \geqslant \tfrac{1}{2}\lambda_3(x)(\Lambda_1(x)/\Lambda_3(x)), \tag{30}$$

and (30) and (24) show that

$$(\lambda_1(x)-\lambda_1(x_1^*))^{1-\kappa_1}\frac{f_1^{\kappa_3-1}(x)}{\varphi_1(x_1^*)} \leqslant 2e^{32}\varphi_1^{-\kappa_1}(x)(\lambda_3'(x)\,\Lambda_1(x))^{\kappa_3-\kappa_1}. \tag{31}$$

A short calculation shows that

$$\begin{aligned}
&\frac{d}{dt}\{\cdots\}\\
&=\{\cdots\}\left(\frac{\lambda_3'(t)}{\lambda_3(x)-\lambda_3(t)}\left\langle(1-\kappa_3)-(1-\kappa_1)\,\varphi_1(t)\frac{\lambda_3(x)-\lambda_3(t)}{\lambda_1(x)-\lambda_1(t)}\right\rangle+\varphi_1\left(\frac{1}{\varphi_1}\right)'\right).
\end{aligned}$$

It follows from $\lambda_3\varphi_1 = (\Lambda_3/\Lambda_1)\,\lambda_1 \uparrow$ that

$$\frac{\varphi_1(t)}{\varphi_1(y)} \leqslant \frac{\lambda_3(y)}{\lambda_3(t)} \leqslant \frac{\lambda_3(y)}{\lambda_3(\bar{x})} \leqslant 2, \qquad \text{if } \bar{x} \leqslant t \leqslant y \leqslant x,$$

and this shows that

$$|\langle\cdots\rangle| = \left|(1-\kappa_3) - (1-\kappa_1)\frac{\varphi_1(t)}{\varphi_1(\xi)}\right| \leqslant 3.$$

Furthermore, $\{\cdots\} \leqslant \lambda_1^{1-\kappa_1}(x)(\lambda_3(x) - \lambda_3(t))^{\kappa_3-1}(1/\varphi_1(t))$ and it follows from (6) that

$$\left|\frac{d}{dt}\{\cdots\}\right| \leqslant 3\lambda_1^{1-\kappa_1}(x)(\lambda_3(x) - \lambda_3(t))^{\kappa_3-1}\frac{\lambda_3'(t)}{\lambda_1(t)}\left(\frac{\Lambda_1(t)}{\Lambda_3(t)}\frac{\lambda_3(t)}{\lambda_3(x)-\lambda_3(t)} + 1\right).$$

We wish to show that $\psi(t) = (\Lambda_1(t)/\Lambda_3(t))\,\lambda_3(t) \leqslant 4(\lambda_3(x) - \lambda_3(t))$, for $\bar{x} \leqslant t \leqslant x_1^*$, and we observe that $\psi(x) \leqslant 2f_1(x) \leqslant 2(\lambda_3(x) - \lambda_3(t))$ by (30). Hence, we need only discuss the case $\psi\downarrow$ and $\lambda_3(x) - \lambda_3(t) \leqslant \psi(t)$, say. It follows from $0 < \tilde{\psi}(\tau) = \psi(\bar{\lambda}_3(\tau))\downarrow$ that $|\tilde{\psi}'| \leqslant \frac{1}{2}$, and then (for that t)

$$\psi(t) - \psi(x) = \tilde{\psi}(\lambda_3(t)) - \tilde{\psi}(\lambda_3(x)) \leqslant \tfrac{1}{2}(\lambda_3(x) - \lambda_3(t)) \leqslant \tfrac{1}{2}\psi(t),$$

i.e.,

$$\psi(t) \leqslant 2\psi(x) \leqslant 4(\lambda_3(x) - \lambda_3(t)).$$

Using this result on ψ we have

$$\left|\frac{d}{dt}\{\cdots\}\right| \leqslant 15\lambda_1^{1-\kappa_1}(x)(\lambda_3(x) - \lambda_3(t))^{\kappa_3-1}\frac{\lambda_3'(t)}{\lambda_1(t)} \qquad \text{for} \quad \bar{x} \leqslant t \leqslant x_1^*. \tag{32}$$

It follows from (31) and (32) that

$$|I_2| \leqslant 2e^{32}\varphi_1^{-\kappa_1}(x)(\lambda_3'(x)\,\Lambda_1(x))^{\kappa_3-\kappa_1}(|A_{\lambda_1}^{\kappa_1}(x, x_1^*)| + |A_{\lambda_1}^{\kappa_1}(x, \bar{x})|)$$
$$+ 15\lambda_1^{1-\kappa_1}(x)\int_{\bar{x}}^{x_1^*}(\lambda_3(x) - \lambda_3(t))^{\kappa_3-1}\frac{\lambda_3'(t)}{\lambda_1(t)}(|A_{\lambda_1}^{\kappa_1}(x,t)| + |A_{\lambda_1}^{\kappa_1}(x,\bar{x})|)\,dt,$$

and it follows from (21) (with $\eta = x$ or $\eta = t$) that

$$|I_2| \leqslant 4e^{32}\left(\lambda_3^{\kappa_3}\frac{V_1}{\lambda_1^{\kappa_1}}\left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_3} + \int_0^x(\lambda_3(x) - \lambda_3(t))^{\kappa_3-1}\lambda_3'(t)\frac{V_1(t)}{\lambda_1^{\kappa_1}(t)}\,dt\right).$$

THEOREM *T*. *Suppose that* (17_1) *holds, that*

$$0 < \kappa_3 \leqslant \kappa_1 \leqslant 1, \qquad A \in M, \qquad \Lambda_1 \dot{\leqslant} \Lambda_3,$$

and that λ_2, V_2 *and* V *satisfy*

$$\Lambda_2^{\kappa_1-\kappa_3} \sim \Lambda_1^{\kappa_1}V^{-1}\Lambda_3^{-\kappa_3}, \qquad \Lambda_1 \dot{\geqslant} \Lambda_2 \tag{33}$$

and

$$\frac{V_2}{\lambda_2^{\kappa_3}} A_2^{\kappa_1} \sim \frac{V_1}{\lambda_1^{\kappa_1}} A_1^{\kappa_1}. \tag{34}$$

Then there is a numerical constant $K_3 > 0$ *such that* $|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1$, $|A_{\lambda_2}^{\kappa_3}| \dot{\leqslant} V_2$ *imply*

$$|A_{\lambda_3}^{\kappa_3}| \dot{\leqslant} K_3 V_3, \qquad V_3 = \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} V + \int_0^x (\lambda_3(x) - \lambda_3(t))^{\kappa_3 - 1} \lambda_3' \frac{V_1}{\lambda_1^{\kappa_1}} dt, \tag{35}$$

if V_3 *satisfies* (17_3). *If* $A_3 \doteq A_1$, *then the integral in* (35) *may be omitted.*

Proof. Let

$$A_{\lambda_3}^{\kappa_3} = \left(\int_0^{x_1^*} + \int_{x_1^*}^{x_2^*} + \int_{x_2^*}^{x}\right)(\cdots)\, dt = I_1 + I_2 + I_3$$

(note that $x_1^* \leqslant x_2^*$ by (33)). It follows from (33) that $2V \dot{\geqslant} (A_1/A_3)^{\kappa_3}$; therefore, by Theorem 2 (including Remark) we need only discuss I_2 and I_3. In what follows, $c_1, c_2, \ldots$, are numerical constants. Writing

$$I_2 = \int_{x_1^*}^{x_2^*} (\lambda_1(x) - \lambda_1(t))^{\kappa_1 - 1}$$
$$\times \lambda_1'(t)\, A(t) \left(\frac{\lambda_3(x) - \lambda_3(t)}{\lambda_1(x) - \lambda_1(t)}\right)^{\kappa_1 - 1} (\lambda_3(x) - \lambda_3(t))^{\kappa_3 - \kappa_1} \frac{dt}{\varphi_1(t)},$$

we obtain from (25), (15) and (24) an estimate

$$|I_2| \dot{\leqslant} c_1(\lambda_3(x) - \lambda_3(x_2^*))^{\kappa_3 - \kappa_1} \varphi_1^{-\kappa_1}(x) \sup_{0 \leqslant \xi \leqslant x} |A_{\lambda_1}^{\kappa_1}(x, \xi)|.$$

We have $\lambda_3(x) - \lambda_3(x_2^*) = f_2(x) \geqslant \frac{1}{2}\lambda_3(x)(A_2(x)/A_3(x))$ (cf. (30)), hence

$$|I_2| \dot{\leqslant} c_2 \lambda_3^{\kappa_3} \frac{A_1^{\kappa_1}}{A_3^{\kappa_3} A_2^{\kappa_1 - \kappa_3}} \sup_{0 \leqslant \xi \leqslant x} |A_{\lambda_1}^{\kappa_1}(x, \xi)| \frac{1}{\lambda_1^{\kappa_1}},$$

and the required estimate of I_2 follows from (33) and (21) ($\eta = x$). Prior to the discussion of I_3 we note that V_2 satisfies (17_2) (with $\kappa_2 = \kappa_3$) since V_1 satisfies (17_1). This is a consequence of

$$\lambda_2^{1-\kappa_3} V_2 \sim \lambda_2 \frac{V_1}{\lambda_1^{\kappa_1}} \left(\frac{A_1}{A_2}\right)^{\kappa_1} \qquad \text{and} \qquad \lambda_2 \dot{\geqslant} c_3 \lambda_1 .$$

Writing

$$I_3 = \int_{x_2^*}^{x} (\lambda_2(x) - \lambda_2(t))^{\kappa_3 - 1} \lambda_2'(t) A(t) \left(\frac{\lambda_3(x) - \lambda_3(t)}{\lambda_2(x) - \lambda_2(t)}\right)^{\kappa_3 - 1} \frac{dt}{\varphi_2(t)},$$

we obtain from (25) ($a = 1$), (15) and (24) an estimate

$$|I_3| \dot{\leqslant} c_4 \varphi_2^{-\kappa_3}(x) \sup_{0 \leqslant \xi \leqslant x} |A_{\lambda_2}^{\kappa_3}(x, \xi)|,$$

and the required estimate of I_3 follows from (21) ($\eta = x$), (34) and (33).

Remarks on Theorem T.

1. If (34) is replaced by $(V_2/\lambda_2^{\kappa_3}) \Lambda_2^{\kappa_1} \dot{\leqslant} c(V_1/\lambda_1^{\kappa_1}) \Lambda_1^{\kappa_1}$ for some $c \geqslant 1$, then (35) holds with cK_3 in place of K_3.

2. If all the assumptions of Theorem T except $\Lambda_1 \dot{\geqslant} \Lambda_2$ are satisfied, then Theorem A_2 may be used to derive from $|A_{\lambda_2}^{\kappa_3}| \dot{\leqslant} V_2$ an estimate $|A_{\lambda_1}^{\kappa_3}| \dot{\leqslant} \tilde{V}_2$ which can serve as a Tauberian condition (case $\lambda_2 = \lambda_1$, $\tilde{V}_2$ in place of V_2). A short calculation shows that $(\tilde{V}_2/\lambda_1^{\kappa_3}) \Lambda_1^{\kappa_1} \preccurlyeq (V_1/\lambda_1^{\kappa_1}) \Lambda_1^{\kappa_1}$ holds, and we have $V \preccurlyeq (\Lambda_1/\Lambda_3)^{\kappa_3}$. This remark leads to the following corollary of Theorem T:

Suppose that the assumptions of Theorem T with the exception of $\Lambda_1 \dot{\geqslant} \Lambda_2$ are satisfied, and that in addition (17_2) holds. Then $|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1$, $|A_{\lambda_2}^{\kappa_3}| \dot{\leqslant} V_2$ imply (for a numerical constant K_4)

$$|A_{\lambda_3}^{\kappa_3}| \dot{\leqslant} K_4 V_3,$$
$$V_3 = \left(V + \left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_3}\right) \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} + \int_0^x (\lambda_3(x) - \lambda_3(t))^{\kappa_3 - 1} \lambda_3' \frac{V_1}{\lambda_1^{\kappa_1}} dt \tag{36}$$

if V_3 of (36) satisfies (17_3).

3. Using (33) and (34) we can express V by the remaining quantities, and we find

$$\lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} V \sim \lambda_3^{\kappa_3} \left(\frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}\right)^{\kappa_3/\kappa_1} \left(\frac{V_2}{\lambda_2^{\kappa_3}}\right)^{1 - \kappa_3/\kappa_1} \Lambda_3^{-\kappa_3}. \tag{37}$$

4. It would seem from the discussion of Theorem T in the introduction that only the cases $1 \preccurlyeq V \preccurlyeq \Lambda_1^{\kappa_1}/\Lambda_3^{\kappa_3}$ are of interest, since in the other cases (35) would follow from $|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1$ by Theorems C or L. Basically, this is the case when $A \in S$. But when $A \notin S$, $\kappa_3 < \kappa_1$, then Theorems C or L are no longer valid, and in this case Theorem T is also of interest for other functions V.

5. Compared with Theorem T, Theorem C–M restricts itself to the case $\Lambda_2 = \Lambda_1$, and in its proof a term corresponding to I_2 does not appear. Thus, the influence of V_1 and V_2 is not balanced in a maximal way.

3. Combinations of the Abelian and Tauberian Theorems

We wish to use the relations (12) and (13) ($\lambda = V_i/\lambda_i^{\kappa_i}$), and this gives reason to introduce the conditions

$$V_i/\lambda_i^{\kappa_i} > \lambda_3^{\delta_i - 1}, \qquad \text{for some} \quad \delta_i > 0, \tag{38$_i$}$$

$$V_i/\lambda_i^{\kappa_i} < \lambda_3^{\Delta_i}, \qquad \text{for some} \quad \Delta_i\,. \tag{39$_i$}$$

By combining the results of the previous section we first prove Theorems C, L, LC.

Theorem C. *Suppose that* (17_1) *and* (38_1) *hold, that* $\Lambda_1 \dot{\leqslant} \Lambda_3$, *and that either*

$$0 \leqslant \kappa_1 \leqslant \kappa_3 \leqslant 1, \qquad A \in M,$$

or

$$0 \leqslant \kappa_3 < \kappa_1 \leqslant 1, \qquad A \in S, \qquad \Lambda_1^{\kappa_1} \dot{\leqslant} \Lambda_3^{\kappa_3},$$

$$\Lambda_1 \dot{\geqslant} \alpha, \qquad V_1(x+1) \dot{\leqslant} c V_1(x) \qquad (\textit{for constants } \alpha > 0,\ c > 0).$$

Then

$$|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1 \quad \textit{implies} \quad |A_{\lambda_3}^{\kappa_3}| \dot{\leqslant} K_5 V_3\,, \qquad V_3 = \lambda_3^{\kappa_3}(V_1/\lambda_1^{\kappa_1}), \tag{40}$$

where K_5 *depends* (*at most*) *on* κ_3, α, c, δ_1, ϵ_1.

Proof. We may assume that $\kappa_3 > 0$ (use Theorem A_3 if $\kappa_3 = 0$), and we note that V_3 then satisfies (17_3) because of (38_1). If $\kappa_3 = \kappa_1$, then we use Theorem T ($\lambda_2 = \lambda_1$, $V_2 = V_1$), and (40) follows from $V \sim (\Lambda_1/\Lambda_3)^{\kappa_3} \dot{\leqslant} 1$ and (13). This result is the second theorem of consistency, and the case $\kappa_3 > \kappa_1$ follows from a combination of this second theorem of consistency and Theorem A_1.

If $\kappa_3 < \kappa_1$, then $|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1$ implies $|A_{\lambda_1}^{\kappa_3}| \dot{\leqslant} K_1 \lambda_1^{\kappa_3}(V_1/\lambda_1^{\kappa_1})\, \Lambda_1^{\kappa_1-\kappa_3}$ by Theorem A_3, and it follows from this estimate and Theorem A_2 that $|A_{\lambda_2}^{\kappa_3}| \dot{\leqslant} V_2 = 5K_1\lambda_2^{\kappa_3}(V_1/\lambda_1^{\kappa_1})(\Lambda_1^{\kappa_1}/\Lambda_2^{\kappa_3})$ for $\Lambda_2 \dot{\leqslant} \Lambda_1$, and we use this estimate for $\Lambda_2 = \alpha$. We now apply Theorem T and Remark 1 ($V \sim (\Lambda_1^{\kappa_1}/\Lambda_3^{\kappa_3})\, \alpha^{\kappa_3-\kappa_1} \dot{\leqslant} \alpha^{\kappa_3-\kappa_1}$), and (40) follows from (35) and (13).

THEOREM *L*. *Suppose that* (17_1) *and* (38_1) *hold, and that*

$$0 \leqslant \kappa_3 < \kappa_1 \leqslant 1, \quad A \in S, \quad \Lambda_1 \geqslant \alpha, \quad \Lambda_1^{\kappa_1} \dot{\geqslant} \Lambda_3^{\kappa_3}, \quad V_1(x+1) \dot{\leqslant} cV_1(x)$$
$$(\alpha > 0,\ c > 0,\ \textit{constant}).$$

Then

$$| A_{\lambda_1}^{\kappa_1} | \dot{\leqslant} V_1 \quad \textit{implies} \quad | A_{\lambda_3}^{\kappa_3} | \dot{\leqslant} K_6 V_3, \qquad V_3 = \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \frac{\Lambda_1^{\kappa_1}}{\Lambda_3^{\kappa_3}} \tag{41}$$

where K_6 *depends (at most) on* κ_3, α, c, δ_1.

Proof. We may assume that $\kappa_3 > 0$ (use Theorem A_3 if $\kappa_3 = 0$), and we note that V_3 then satisfies (17_3) because of (38_1). If $\Lambda_1 \dot{\leqslant} \Lambda_3$, then (41) follows from Theorem T in exactly the same way as (40) did (we now have $V \asymp \Lambda_1^{\kappa_1}/\Lambda_3^{\kappa_3} \dot{\geqslant} 1$), and (41) follows for $\Lambda_1 \dot{\geqslant} \Lambda_3$ from Theorems A_3, A_2 (cf. the proof of Theorem C).

THEOREM *LC*. *Suppose that* (17_1) *holds, and that*

$$0 \leqslant \kappa_1 \leqslant \kappa_3 \leqslant 1, \qquad A \in M, \qquad \Lambda_3 \dot{\leqslant} \Lambda_1 .$$

Then

$$| A_{\lambda_1}^{\kappa_1} | \dot{\leqslant} V_1 \quad \textit{implies} \quad | A_{\lambda_3}^{\kappa_3} | \dot{\leqslant} K_7 V_3, \qquad V_3 = \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \left(\frac{\Lambda_1}{\Lambda_3} \right)^{\kappa_1}, \tag{42}$$

where K_7 *depends (at most) on* κ_3, ϵ_1.

Proof. We have already pointed out in the introduction, that (42)follows from a combination of Theorems A_1, A_2.

In Theorems C and L we have used the condition (38_1) in order to replace the integral in (35) by $\lambda_3^{\kappa_3}(V_1/\lambda_1^{\kappa_1})$. If also (39_1) holds, then this is sharp by (12), and one expects best estimates. On the other hand, if (39_1) does not hold, then we must retain the integral in (35) if we want sharp results. Theorems A_1, A_2, A_3 and T are general enough to furnish the corresponding results. This remark also applies to the following theorems (where the fact, that no integral appears in (35) whenever $\Lambda_1 \doteq \Lambda_3$ is important in some cases).

If, on the other hand, $V_1/\lambda_1^{\kappa_1}$ is rather small and does not satisfy (38_1), then it follows from (12) that the integral in (35) may be replaced by $\lambda_3^{\kappa_3-1}(x) \int_0^\infty \lambda_3'(V_1/\lambda_1^{\kappa_1})\, dt$, and it is also possible in this case to prove the results corresponding to (40) and (41).

The following theorems are of Tauberian nature.

THEOREM 3. *Suppose that* (17_1), (17_2) *and* (38_1), (38_2) *hold, and that*

$$0 \leqslant \kappa_2 \leqslant \kappa_3 < \kappa_1 \leqslant 1, \qquad A \in M.$$

Then $|A_{\lambda_1}^{\kappa_1}| \dot{\leqslant} V_1$, $|A_{\lambda_2}^{\kappa_2}| \dot{\leqslant} V_2$ *imply* $|A_{\lambda_3}^{\kappa_3}| \leqslant K_8 V_3$,

$$V_3 = \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \left(1 + \left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_3}\right)$$

$$+ \lambda_3^{\kappa_3} \left(\frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}\right)^{(\kappa_3-\kappa_2)/(\kappa_1-\kappa_2)} \left(\frac{V_2}{\lambda_2^{\kappa_2}} \Lambda_2^{\kappa_2}\right)^{(\kappa_1-\kappa_3)/(\kappa_1-\kappa_2)} \Lambda_3^{-\kappa_3}$$

$$+ \lambda_3^{\kappa_3} \left(\frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}\right)^{\kappa_3/\kappa_1} \left(\frac{V_2}{\lambda_2^{\kappa_2}}\right)^{1-\kappa_3/\kappa_1} \Lambda_3^{-\kappa_3}, \tag{43}$$

where K_8 *depends* (*at most*) *on* κ_3, δ_1, δ_2, ϵ_2.

Before we turn to the proof we indicate its main idea by the following diagrams:

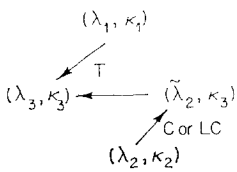

DIAGRAM 6.

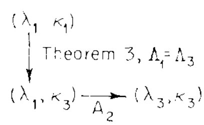

DIAGRAM 7.

If $\Lambda_1 \dot{\leqslant} \Lambda_3$, then we move from (λ_2, κ_2) to $(\tilde{\lambda}_2, \kappa_3)$ with an Abelian Theorem ($\tilde{\lambda}_2$ is determined by (33) and (34)), and then we apply Theorem T. This Abelian Theorem may be C or LC, and we combine both theorems (for this case) into

$$|A_{\tilde{\lambda}_2}^{\kappa_3}| \dot{\leqslant} C\tilde{\lambda}_2^{\kappa_3} \frac{V_2}{\lambda_2^{\kappa_2}} \left(1 + \left(\frac{\Lambda_2}{\tilde{\Lambda}_2}\right)^{\kappa_2}\right) = \tilde{V}_2$$

where $C \geqslant 1$ depends (at most) on κ_3, ϵ_2, δ_2. If $\Lambda_1 \dot{>} \Lambda_3$, then we use the preceding part with $\Lambda_1 \doteq \Lambda_3$ in order to obtain an estimate $|A_{\lambda_1}^{\kappa_3}| \dot{\leqslant} V_1^*$, and we move from this estimate to the estimate of $A_{\lambda_3}^{\kappa_3}$ by Theorem A_2.

Proof of Theorem 3. We may assume that $\kappa_3 > 0$ (for $\kappa_3 = 0$ the third term of (43) is V_2). Assume first that $\Lambda_1 \dot{\leqslant} \Lambda_3$. Let

$$H = \frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1} \Big/ \frac{V_2}{\lambda_2^{\kappa_2}} \Lambda_2^{\kappa_1},$$

and let (cf. footnote 8)

$$\text{(i)}\quad \begin{cases} \tilde{\Lambda}_2 \sim \Lambda_2 H^{1/(\kappa_1-\kappa_2)} & \text{if} \quad H \dot{<} 1, \\ \tilde{\Lambda}_2 = \Lambda_1 & \text{if} \quad H \dot{\geqslant} 1, \quad \Lambda_1 \dot{\leqslant} \Lambda_2, \end{cases}$$

$$\text{(ii)}\quad \tilde{\Lambda}_2 \sim \min(\Lambda_2 H^{1/\kappa_1}, \Lambda_1) \qquad \text{if} \quad H \dot{\geqslant} 1, \quad \Lambda_1 \dot{>} \Lambda_2 .$$

In case (i) we have $\Lambda_2/\tilde{\Lambda}_2 \dot{\geqslant} \frac{1}{2}$, $\tilde{\Lambda}_2^{\kappa_1-\kappa_2} \dot{\leqslant} 2\Lambda_2^{\kappa_1-\kappa_2}H$, and in case (ii) we have $\Lambda_2/\tilde{\Lambda}_2 \dot{\leqslant} 2$, $\tilde{\Lambda}_2^{\kappa_1} \dot{\leqslant} 2\Lambda_2^{\kappa_1}H$.

In case (i) we have

$$\frac{\tilde{V}_2}{\tilde{\lambda}_2^{\kappa_3}} \tilde{\Lambda}_2^{\kappa_1} = C \frac{V_2}{\lambda_2^{\kappa_2}} \tilde{\Lambda}_2^{\kappa_1} \left(1 + \left(\frac{\Lambda_2}{\tilde{\Lambda}_2}\right)^{\kappa_2}\right)$$

$$\dot{\leqslant} 3C \frac{V_2}{\lambda_2^{\kappa_2}} \tilde{\Lambda}_2^{\kappa_1-\kappa_2} \Lambda_2^{\kappa_2} \dot{\leqslant} 6C \frac{V_2}{\lambda_2^{\kappa_2}} \Lambda_2^{\kappa_1} H = 6C \frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}$$

and (36) ($V = \Lambda_1^{\kappa_1} \Lambda_3^{-\kappa_3} \Lambda_2^{\kappa_3-\kappa_1} H^{(\kappa_3-\kappa_1)/(\kappa_1-\kappa_2)}$ or $V = (\Lambda_1/\Lambda_3)^{\kappa_3}$) and (13) yield the first and second term of (43).

In case (ii) we have

$$\frac{\tilde{V}_2}{\tilde{\lambda}_2^{\kappa_3}} \tilde{\Lambda}_2^{\kappa_1} \dot{\leqslant} 3C \frac{V_2}{\lambda_2^{\kappa_2}} \tilde{\Lambda}_2^{\kappa_1} \dot{\leqslant} 6C \frac{V_2}{\lambda_2^{\kappa_2}} \Lambda_2^{\kappa_1} H = 6C \frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}$$

and (36) and (13) yield the first and third term of (43).

If $\Lambda_1 \dot{>} \Lambda_3$, then it follows from the part of Theorem 3 which has already been proven that $| A_{\lambda_1}^{\kappa_3} | \dot{\leqslant} V_1^*$, where V_1^* is V_3 of (43) with $\Lambda_3 = \Lambda_1$. We apply Theorem A_2 (note that V_1^* satisfies (17_1)) and obtain $| A_{\lambda_3}^{\kappa_3} | \dot{\leqslant} 5\lambda_3^{\kappa_3}(V_1^*/\lambda_1^{\kappa_3})(\Lambda_1/\Lambda_3)^{\kappa_3}$, and this proves (43).

THEOREM 4. *Suppose that* (17_1), (17_2) *and* (38_1), (38_2) *hold, and that*

$$0 \leqslant \kappa_3 < \kappa_1 \leqslant 1, \quad \kappa_3 < \kappa_2 \leqslant 1, \quad A \in S, \quad \Lambda_2 \dot{\geqslant} \alpha, \quad V_2(x+1) \dot{\leqslant} cV_2(x)$$
$$(\alpha > 0,\ c > 0,\ \textit{constant}).$$

Then $| A_{\lambda_1}^{\kappa_1} | \dot{\leqslant} V_1$, $| A_{\lambda_2}^{\kappa_2} | \dot{\leqslant} V_2$ *imply* $| A_{\lambda_3}^{\kappa_3} | \dot{\leqslant} K_9 V_3$,

$$V_3 = \lambda_3^{\kappa_3} \frac{V_1}{\lambda_1^{\kappa_1}} \left(1 + \left(\frac{\Lambda_1}{\Lambda_3}\right)^{\kappa_3}\right) + \lambda_3^{\kappa_3} \left(\frac{V_2}{\lambda_2^{\kappa_2}} \Lambda_2^{\kappa_2}\right) \Lambda_3^{-\kappa_3}$$
$$+ \lambda_3^{\kappa_3} \left(\frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}\right)^{\kappa_3/\kappa_1} \left(\frac{V_2}{\lambda_2^{\kappa_2}}\right)^{1-\kappa_3/\kappa_1} \Lambda_3^{-\kappa_3}, \tag{44}$$

where K_9 *depends (at most) on* κ_3, α, c, δ_2.

The idea of the proof is in principle the same as in Theorem 3, and it is (for $\Lambda_1 \dot{\leqslant} \Lambda_3$) indicated in the following diagram:

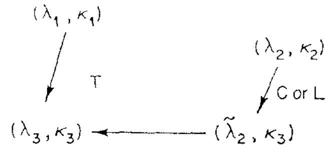

DIAGRAM 8.

We combine the Abelian Theorems C and L (for this case) into

$$| A^{\kappa_3}_{\tilde{\lambda}_2} | \leqslant D\tilde{\lambda}_2^{\kappa_3} \frac{V_2}{\lambda_2^{\kappa_2}} \left(1 + \frac{\Lambda_2^{\kappa_2}}{\tilde{\Lambda}_2^{\kappa_3}}\right) = \tilde{V}_2 ,$$

where $D \geqslant 1$ depends (at most) on κ_3, α, c, δ_2.

Proof of Theorem 4. We may assume that $\kappa_3 > 0$. Assume first that $\Lambda_1 \dot{\leqslant} \Lambda_3$, and let $H^* = H\Lambda_2^{(\kappa_1/\kappa_3)(\kappa_3-\kappa_2)}$ (H as in the proof of Theorem 3). Let

$$\text{(i)} \quad \begin{cases} \tilde{\Lambda}_2^{\kappa_3} \sim \Lambda_2^{\kappa_2} H^{*\kappa_3/(\kappa_1-\kappa_3)}, & \text{if } H^* \dot{<} 1, \\ \tilde{\Lambda}_2 = \Lambda_1, & \text{if } H^* \dot{\geqslant} 1, \quad \Lambda_1^{\kappa_3} \dot{\leqslant} \Lambda_2^{\kappa_2}, \end{cases}$$

$$\text{(ii)} \quad \tilde{\Lambda}_2^{\kappa_3} \sim \min(\Lambda_2^{\kappa_2} H^{*\kappa_3/\kappa_1}, \Lambda_1^{\kappa_3}) \qquad \text{if } H^* \dot{\geqslant} 1, \quad \Lambda_1^{\kappa_3} \dot{>} \Lambda_2^{\kappa_2}.$$

(Cf. footnote (8); because of Theorem L we may assume $\tilde{\Lambda}_2 \dot{\leqslant} \Lambda_3$ if $H^* \dot{<} 1$.) In case (i) we have $\Lambda_2^{\kappa_2}/\tilde{\Lambda}_2^{\kappa_3} \dot{\geqslant} \frac{1}{2}$, $\tilde{\Lambda}_2^{\kappa_1-\kappa_3} \dot{\leqslant} 2\Lambda_2^{(\kappa_2/\kappa_3)(\kappa_1-\kappa_3)}H^*$, and in case (ii) we have $\Lambda_2^{\kappa_2}/\tilde{\Lambda}_3^{\kappa_3} \dot{\leqslant} 2$, $\tilde{\Lambda}_2^{\kappa_3} \dot{\leqslant} 2\Lambda_2^{\kappa_2}H^{*\kappa_3/\kappa_1}$. We proceed as in the proof of Theorem 3. In case (i) we have

$$\begin{aligned} \frac{\tilde{V}_2}{\tilde{\lambda}_2^{\kappa_3}} \tilde{\Lambda}_2^{\kappa_1} &= D \frac{V_2}{\lambda_2^{\kappa_2}} \tilde{\Lambda}_2^{\kappa_1} \left(1 + \frac{\Lambda_2^{\kappa_2}}{\tilde{\Lambda}_2^{\kappa_3}}\right) \\ &\dot{\leqslant} 3D \frac{V_2}{\lambda_2^{\kappa_2}} \tilde{\Lambda}_2^{\kappa_1-\kappa_3} \Lambda_2^{\kappa_2} \dot{\leqslant} 6D \frac{V_2}{\lambda_2^{\kappa_2}} \Lambda_2^{\kappa_1} H = 6D \frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1}, \end{aligned}$$

and in case (ii) we have

$$\frac{\tilde{V}_2}{\tilde{\lambda}_2^{\kappa_3}} \tilde{\Lambda}_2^{\kappa_1} \dot{\leqslant} 3D \frac{V_2}{\lambda_2^{\kappa_2}} \tilde{\Lambda}_2^{\kappa_1} \dot{\leqslant} 6D \frac{V_2}{\lambda_2^{\kappa_2}} \Lambda_2^{\kappa_1} H = 6D \frac{V_1}{\lambda_1^{\kappa_1}} \Lambda_1^{\kappa_1},$$

and (36) and (13) yield (44). The case $\Lambda_1 \dot{>} \Lambda_3$ follows from this result as in the proof of Theorem 3.

4. The Main Theorem

In order to simplify the formulas in Theorem 5, we introduce some abbreviations.

Denoting by $i, j, i \neq j$, the subscripts 1, 2, we define:

$$C_i = \frac{V_i}{\lambda_i^{\kappa_i}} \Lambda_3^{\kappa_3}, \qquad L_i = \frac{V_i}{\lambda_i^{\kappa_i}} \Lambda_i^{\kappa_i},$$

$$LC_i = \frac{V_i}{\lambda_i^{\kappa_i}} \left(\frac{\Lambda_i}{\Lambda_3}\right)^{\kappa_i} \Lambda_3^{\kappa_3}, \qquad TCL_i = \frac{V_i}{\lambda_i^{\kappa_i}} \Lambda_i^{\kappa_3},$$

$$K_i^{\mathrm{I}} = (L_i)^{(\kappa_3-\kappa_j)/(\kappa_i-\kappa_j)} (L_j)^{(\kappa_i-\kappa_3)/(\kappa_i-\kappa_j)}, \qquad 0 \leqslant \kappa_j \leqslant \kappa_3 < \kappa_i \leqslant 1,$$

$$K_i^{\mathrm{II}} = (LC_i)^{\kappa_3/\kappa_i} (C_j)^{1-\kappa_3/\kappa_i}, \qquad 0 \leqslant \kappa_3 < \kappa_i \leqslant 1,$$

$$A_i = (C_i + L_i + LC_i)\, \lambda_3'^{\kappa_3},$$

$$T_i^{\mathrm{I}} = (C_i + TCL_i + K_i^{\mathrm{I}} + K_i^{\mathrm{II}})\, \lambda_3'^{\kappa_3},$$

$$T_i^{\mathrm{II}} = (C_i + TCL_i + L_j + K_i^{\mathrm{II}})\, \lambda_3'^{\kappa_3}$$

In Theorem 5 only A_i, T_i^{I}, T_i^{II} appear, and these quantities are built from C_i, L_i, LC_i, TCL_i. When multiplied by $\lambda_3'^{\kappa_3}$, the terms C_1, L_1, LC_1 are the V_3's of Theorems C, L and LC, and A_1 is the corresponding V_3 in the combination of all Abelian theorems (see footnote 6). The expression $TCL_1\lambda_3'^{\kappa_3}$ results when Theorem C, extended by Theorem T to $\Lambda_3 \geqslant \Lambda_1$ (see the introduction), is applied and then followed by Theorem L (similarly to LC_1). The expressions K^{I}, K^{II} are "convex" combinations of L's or LC's and C's.

Theorem 5. *Suppose that* (17_1), (17_2), (38_1), (38_2) *hold, that*

$$V_1(x+1) \asymp V_1(x), \quad V_2(x+1) \asymp V_2(x), \quad \Lambda_1 \geqslant 1, \quad \Lambda_2 \geqslant 1, \quad \Lambda_3 \geqslant 1,$$

and that $A \in M$. *Then* $A_{\lambda_1}^{\kappa_1} \leqslant V_1$, $A_{\lambda_2}^{\kappa_2} \leqslant V_2$ *imply* $A_{\lambda_3}^{\kappa_3} \leqslant V_3$, *where*

$$V_3 = \min(A_1, A_2), \qquad \textit{if} \quad 0 \leqslant \kappa_1 \leqslant \kappa_3, \quad 0 \leqslant \kappa_2 \leqslant \kappa_3 \leqslant 1,$$

$$V_3 = \min(A_i, T_j^{\mathrm{I}}), \qquad \textit{if} \quad 0 \leqslant \kappa_i \leqslant \kappa_3 < \kappa_j \leqslant 1.$$

If, in addition, $A \in S$, *then*

$$V_3 = \min(A_1, A_2, T_j^{\mathrm{I}}), \qquad \textit{if} \quad 0 \leqslant \kappa_i \leqslant \kappa_3 < \kappa_j \leqslant 1,$$

$$V_3 = \min(A_1, A_2, T_1^{\mathrm{II}}, T_2^{\mathrm{II}}), \qquad \textit{if} \quad 0 \leqslant \kappa_3 < \kappa_1 \leqslant 1, \quad \kappa_3 < \kappa_2 \leqslant 1.$$

These functions V_3 *are minimal bounds whenever* (39_1), (39_2) *holds. Let the dependency of the function* $V_3 = V_3(x)$ *upon* $V_1 = V_1(x)$, $V_2 = V_2(x)$ *be indicated by* $V_3 = V_3[V_1, V_2]$. *If* $V_3[\epsilon V_1, V_2] \dot{\leqslant} \gamma_1 \epsilon^{\gamma_2} V_3[V_1, V_2]$ *holds for* $0 < \epsilon \leqslant 1$ *with fixed* $\gamma_{1,2} > 0$, *then*

$$A_{\lambda_1}^{\kappa_1} \prec V_1, \qquad A_{\lambda_2}^{\kappa_2} \preccurlyeq V_2 \quad \textit{imply} \quad A_{\lambda_3}^{\kappa_3} \prec V_3.$$

Similarly, if $V_3[V_1, \epsilon V_2] \dot{\leqslant} \gamma_1 \epsilon^{\gamma_2} V_3[V_1, V_2]$, *then*

$$A_{\lambda_1}^{\kappa_1} \preccurlyeq V_1, \qquad A_{\lambda_2}^{\kappa_2} \prec V_2 \quad \textit{imply} \quad A_{\lambda_3}^{\kappa_3} \prec V_3.$$

Theorems C, L, LC, 3 and 4 show that the estimates $A_{\lambda_3}^{\kappa_3} \preccurlyeq V_3$ of Theorem 5 are true, and the statements concerning $\prec$ also follow from these theorems. It remains only to show that Theorem 5 gives minimal bounds, and the rest of this section is devoted to this proof.

Let V_3 be one of the functions which appear in Theorem 5, and suppose that $U(x)$ is nonnegative on $(0, \infty)$, and that $V_3 \not\preccurlyeq U$. Then V_3 is minimal, if we can find $A \in M$ or $A \in S$ such that $A_{\lambda_1}^{\kappa_1} \preccurlyeq V_1$, $A_{\lambda_2}^{\kappa_2} \preccurlyeq V_2$ and $A_{\lambda_3}^{\kappa_3} \not\preccurlyeq U$. In the following we will first give the general construction of such A's, and then we will apply it to the individual functions V_3.

If $V_3 \not\preccurlyeq U$, then we can find a sequence $0 < x_n' \uparrow \infty$ such that $U(x_n')/V_3(x_n') \to 0$. In view of (17_i) there is a subsequence $\{x_n\}$ of $\{x_n'\}$ such that

$$V_i(x_{n-1})\, \lambda_i^{1-\kappa_i}(x_{n-1}) \dot{\leqslant} \tfrac{1}{2} V_i(x_n)\, \lambda_i^{1-\kappa_i}(x_n), \tag{45}$$

$$\lambda_i(x_{n-1}) \leqslant \tfrac{1}{2}\lambda_i(\bar{\lambda}_3(\tfrac{1}{2}\lambda_3(x_n))), \qquad i = 1, 2, 3. \tag{46}$$

Let $0 < f(x) \leqslant \frac{1}{2}\lambda_3(x), f \in L$, $g_i(x) = V_i(x) \max(\lambda_i^{-\kappa_i}(x), (f(x)(\lambda_i'(x)/\lambda_3'(x))^{-\kappa_i}))$, $g(x) = \min(g_1(x), g_2(x))$, $z(x) = \bar{\lambda}_3(\lambda_3(x) - f(x))$, $z_n = z(x_n)$.

LEMMA 8. *Suppose that* (17_1), (17_2) *hold, that* $0 \leqslant k_\nu \leqslant 1$, $(\nu = 1, 2, 3)$ *and that*

$$g(x) \preccurlyeq g(t) \qquad \textit{if} \quad z(x) \leqslant t \leqslant x. \tag{47}$$

Let

$$A(t) = \begin{cases} g(x_n), & \textit{if} \quad z_n \leqslant t \leqslant x_n \\ 0, & \textit{otherwise}^{18}. \end{cases}$$

Then $A_{\lambda_1}^{\kappa_1}(x) \preccurlyeq V_1(x)$, $A_{\lambda_2}^{\kappa_2}(x) \preccurlyeq V_2(x)$, $A_{\lambda_3}^{\kappa_3}(x_n) \succcurlyeq g(x_n) f^{\kappa_3}(x_n)$.

[18] We have $x_{n-1} \leqslant \bar{\lambda}_3(\frac{1}{4}\lambda_3(x_n)) < \bar{\lambda}_3(\lambda_3(x_n) - f(x_n))$ by (46), $i = 3$.

Proof. The statement on $A_{\lambda_3}^{\kappa_3}$ follows for $\kappa_3 > 0$ ($\kappa_3 = 0$ is trivial) from

$$A_{\lambda_3}^{\kappa_3}(x_n) \geqslant g(x_n) \int_{z_n}^{x_n} (\lambda_3(x_n) - \lambda_3(t))^{\kappa_3 - 1} \lambda_3'(t)\, dt \asymp g(x_n) f^{\kappa_3}(x_n).$$

Let i be 1 or 2, and let first $z_n \leqslant x \leqslant x_n$. It follows from Lemma 3 that

$$f(x) \preccurlyeq f(t), \qquad \text{if} \quad z(x) \leqslant t \leqslant x, \tag{48}$$

and it follows from (46) and $x \geqslant z_n \geqslant \bar{x}_n$ that

$$\frac{\lambda_i(x_{n-1})}{\lambda_i(x)} \leqslant \frac{\lambda_i(x_{n-1})}{\lambda_i(\bar{\lambda}_3(\frac{1}{2}\lambda_3(x_n)))} \leqslant \frac{1}{2},$$

in particular

$$\lambda_i(x) - \lambda_i(x_{n-1}) \asymp \lambda_i(x). \tag{49}$$

If $\kappa_i = 0$, then $A_{\lambda_i}^{\kappa_i}(x) \preccurlyeq V_i(x)$ by (47), hence we may assume that $\kappa_i > 0$. We have, by Lemma 6, (10), and (49),

$$\begin{aligned} A_{\lambda_i}^{\kappa_i}(x) &\leqslant \int_{z_n}^{x} g(x_n)(\lambda_i(x) - \lambda_i(t))^{\kappa_i - 1} \lambda_i'(t)\, dt \\ &\quad + \sum_{\nu=1}^{n-1} g(x_\nu)(\lambda_i(x) - \lambda_i(x_{n-1}))^{\kappa_i - 1} \int_{z_\nu}^{x_\nu} \lambda_i'(t)\, dt \\ &\asymp g(x_n) \min\left(\lambda_i^{\kappa_i}(x), \left((f(x_n) + \lambda_3(x) - \lambda_3(x_n)) \frac{\lambda_i'(x)}{\lambda_3'(x)}\right)^{\kappa_i}\right) \\ &\quad + (\lambda_i(x))^{\kappa_i - 1} \sum_{\nu=1}^{n-1} g(x_\nu) \min\left(\lambda_i(x_\nu), f(x_\nu) \frac{\lambda_i'(x_\nu)}{\lambda_3'(x_\nu)}\right) \\ &\preccurlyeq g(x_n) \min\left(\lambda_i^{\kappa_i}(x), \left(f(x_n) \frac{\lambda_i'(x)}{\lambda_3'(x)}\right)^{\kappa_i}\right) \\ &\quad + \lambda_i^{\kappa_i - 1}(x) \sum_{\nu=1}^{n-1} g(x_\nu) \min\left(\lambda_i^{\kappa_i}(x_\nu), \left(f(x_\nu) \frac{\lambda_i'(x_\nu)}{\lambda_3'(x_\nu)}\right)^{\kappa_i}\right) \lambda_i^{1-\kappa_i}(x_\nu). \end{aligned}$$

It follows from (47), (48) and the definition of g that

$$A_{\lambda_i}^{\kappa_i}(x) \preccurlyeq V_i(x) + \lambda_i^{\kappa_i - 1}(x) \sum_{\nu=1}^{n-1} V_i(x_\nu)\, \lambda_i^{1-\kappa_i}(x_\nu),$$

and $A_\lambda^{\kappa_i} \preccurlyeq V_i$ follows from (17_i) and (45).

Next, let $x_{n-1} < x < z_{n-1}$ and $\kappa_i > 0$. It follows from Theorem 1 and from the definition of A that

$$A_{\lambda_i}^{\kappa_i}(x) = A_{\lambda_i}^{\kappa_i}(x, x_{n-1}) = \lambda_i^{\kappa_i - 1}(x)\, A_{\lambda_i}^{\kappa_i}(\xi_0)\, \lambda_i^{1-\kappa_i}(\xi_0), \qquad 0 \leqslant \xi_0 \leqslant x_{n-1}.$$

If $z_\nu \leqslant \xi_0 \leqslant x_\nu$ for some $\nu \leqslant n-1$, then $A_{\lambda_i}^{\kappa_i}(x) \preccurlyeq V_i(x)$ follows from (17_i) and the previous part of the proof.

If $x_{\nu-1} < \xi_0 < z_\nu$ for some $\nu \leqslant n-1$, then

$$A_{\lambda_i}^{\kappa_i}(\xi_0) = A_{\lambda_i}^{\kappa_i}(\xi_0, x_{\nu-1}) = \lambda_i^{\kappa_i-1}(\xi_0)\, A_{\lambda_i}^{\kappa_i}(\xi_1)\, \lambda_i^{1-\kappa_i}(\xi_1), \qquad \xi_1 \leqslant x_{\nu-1},$$

i.e.,

$$A_{\lambda_i}^{\kappa_i}(x) = \lambda_i^{\kappa_i-1}(x)\, A_{\lambda_i}^{\kappa_i}(\xi_1)\, \lambda_i^{1-\kappa_i}(\xi_1), \qquad \xi_1 \leqslant x_{n-2},$$

and we proceed as before (with ξ_1 in place of ξ_0). After at most n steps we obtain $A_{\lambda_i}^{\kappa_i}(x) \preccurlyeq V_i(x)$ (observe (17_i)).

This proof also shows that in case $f(x) \succ \lambda_3'(x)$, in which $x_n - z_n > 1$, the definition of A can be changed (slightly) to ensure $A \in S$ by using $[x_n]+1$, $[z_n]$ instead of x_n, z_n. In case $f(x_n) \asymp \lambda_3(x_n) - \lambda_3(x_n - 1) \asymp \lambda_3'(x_n)$ we replace x_n by $[x_n]+1$ and z_n by $[x_n]-1$, and change (46) to

$$\lambda_i([x_{n-1}]+1) \leqslant \frac{1}{2}\lambda_i\left(\frac{[x_n]+1}{2}\right), \qquad i = 1, 2, 3. \tag{50}$$

Thus, if $f(x) \succcurlyeq \lambda_3'(x)$ Lemma 8 remains true with $A \in S$.

In order to apply the construction to the individual V_3's of Theorem 5 we must find for each V_3 a function f such that $g(x_n) f^{\kappa_3}(x_n) \asymp V_3(x_n)$. If f and g satisfy the requirements of Lemma 8, then $A_{\lambda_1}^{\kappa_1} \preccurlyeq V_1$, $A_{\lambda_2}^{\kappa_2} \preccurlyeq V_2$ but

$$A_{\lambda_3}^{\kappa_3}(x_n) \succcurlyeq V_3(x_n) \succ U(x_n).$$

We choose f according to the leading term occurring in V_3 (i.e., C, LC, L, K^{I}, K^{II}). In this context we observe that TCL need not be used since it never is the only leading term. In order to facilitate the calculations we split these four cases into eight cases as follows (i, j, $i \neq j$ are 1 and 2):

1. $V_3\lambda_3'^{-\kappa_3} \asymp C_i \preccurlyeq C_j, \qquad \Lambda_1 \preccurlyeq \Lambda_3, \qquad \Lambda_2 \preccurlyeq \Lambda_3;$

2. $V_3\lambda_3'^{-\kappa_3} \asymp C_i \preccurlyeq LC_j \qquad \Lambda_i \preccurlyeq \Lambda_3 \preccurlyeq \Lambda_j;$

3. $V_3\lambda_3'^{-\kappa_3} \asymp LC_i \preccurlyeq C_j, \qquad \Lambda_j \preccurlyeq \Lambda_3 \preccurlyeq \Lambda_i;$

4. $V_3\lambda_3'^{-\kappa_3} \asymp LC_i \preccurlyeq LC_j, \qquad \Lambda_3 \preccurlyeq \Lambda_1, \qquad \Lambda_3 \preccurlyeq \Lambda_2;$

5. $V_3\lambda_3'^{-\kappa_3} \asymp L_i \preccurlyeq L_j;$

6. $V_3\lambda_3'^{-\kappa_3} \asymp K_i^{\text{II}}$, $\quad 1 \leqslant \Lambda_j \leqslant \Lambda_j H_i^{1/\kappa_i} \leqslant \Lambda_i$, $\quad \Lambda_j H_i^{1/\kappa_i} \leqslant \Lambda_3$,

$$H_i = \frac{\Lambda_3^{\kappa_3} L_i}{\Lambda_j^{\kappa_i} C_j};$$

7. $V_3\lambda_3'^{-\kappa_3} \asymp K_i^{\text{I}}$, $\quad 1 \leqslant \Lambda_j H_i^{1/(\kappa_i-\kappa_j)} \leqslant \Lambda_\nu \quad (\nu = 1, 2, 3)$;

8. $V_3\lambda_3'^{-\kappa_3} \asymp K_i^{\text{I}}$, $\quad \Lambda_j H_i^{1/(\kappa_i-\kappa_j)} \prec 1$ and not case "$\Lambda \in S$."

(Compare cases 6, 7 and 8 with the proof of Theorem 3.)

Our claim is that every individual case of Theorem 5 is contained in at least one of these eight cases. In the simplest case of Theorem 5, viz. $0 \leqslant \kappa_1 \leqslant \kappa_3$, $0 \leqslant \kappa_2 \leqslant \kappa_3 \leqslant 1$, we find that $V_3\lambda_3'^{-\kappa_3}$ is given by C_i or LC_i and that only the cases 1, 2 resp. 3, 4 are possible. The discussion of all other cases is rather lengthy, but represents no difficulty and is, therefore, omitted.

If

$$f(x) = \tfrac{1}{2}\lambda_3(x), \qquad \text{in cases 1, 2, 3, 4,}$$

$$f(x) = K\lambda_3'(x), \qquad \text{in case 5,}$$

$$f(x) = K\Lambda_j H_i^{1/\kappa_i}\lambda_3'(x), \qquad \text{in case 6,}$$

$$f(x) = K\Lambda_j H_i^{1/(\kappa_i-\kappa_j)}\lambda_3'(x), \text{ in cases 7 and 8,}$$

where K in each case is chosen such that $f(x) \leqslant \frac{1}{2}\lambda_3(x)$ (observe that $\Lambda_3 \geqslant 1$), then the relation involving V_3 in cases 1–8 is satisfied, and it only remains to show that (47) holds. This can be done as follows.

Observe first that $g = g_i$ for $i = 1$ or 2. Next, observe that

$$V_i(t)/\lambda_i^{\kappa_i}(t) \asymp V_i(x)/\lambda_i^{\kappa_i}(x), \tag{51}$$

by (38_i), (39_i) and Lemma 3. Also (by Lemma 3)

$$\frac{\Lambda_i(x)}{\Lambda_3(x)}\,\lambda_3(x) \leqslant \frac{\Lambda_i(t)}{\Lambda_3(t)}\,\lambda_3(t), \tag{52}$$

since $(\Lambda_i(x)/\Lambda_3(x))\,\lambda_3(x) \leqslant \lambda_3^2(x)$. This shows that (47) holds when $f(x) = \frac{1}{2}\lambda_3(x)$. If $f = K\lambda_3'(x)$, then

$$f(x) = K\lambda_3'(x) \leqslant \lambda_3(x)\min\left(\frac{1}{2}, c\,\frac{|\lambda_3'(x)/\lambda_3''(x)|}{\Lambda_3(x)}\right), \qquad c > 0^{19}.$$

[19] Observe that $\Lambda_3 \geqslant 1$, and this implies $\lambda_3'/\lambda_3'' \leqslant \lambda_3/\lambda_3'$.

It follows from Lemma 2 ($\lambda = \lambda_3'$) and the Remark after Lemma 2 that $\lambda_3'(t) \asymp \lambda_3'(x)$ if $0 \leqslant \lambda_3(x) - \lambda_3(t) \leqslant f(x)$, i.e., $z(x) \leqslant t \leqslant x$. This shows that (47) holds when $f = K\lambda_3'(x)$.

If $f(x) = KA_jH_i^{1/\kappa_i}\lambda_3'(x)$, then $g \asymp V_j/\lambda_j^{\kappa_j}$, and (47) follows from (51).

Finally, if $f = KA_jH_i^{1/(\kappa_i-\kappa_j)}\lambda_3'$, then $f(x) \leqslant \lambda_3(x)\min(\frac{1}{2}, c(A_\nu(x)/A_3(x)))$, $\nu = 1, 2, 3$. We have

$$f(t) \asymp \left(\frac{V_i(t)/\lambda_i^{\kappa_i}(t)}{V_j(t)/\lambda_j^{\kappa_j}(t)} \frac{(\lambda_3'(t)\,A_i(t))^{\kappa_i}}{(\lambda_3'(t)\,A_j(t))^{\kappa_j}}\right)^{1/(\kappa_i-\kappa_j)} \asymp f(x)$$

by Lemma 2, (9) and (51). Hence, (47) holds in all cases.

5. Concluding Remarks

If (38_i) does not hold, i.e., if $V_i/\lambda_i^{\kappa_i}$ is rather small, then the discussion after Theorem *LC* indicates how to modify the definition of V_3 so that Theorem 5 remains true. The essential point is to treat the integral in (35) correctly, if it occurs at all. Furthermore, an analysis of Theorem 2 shows that the integral in (28) may not be optimal if $A_1 \sim A_3$ (due to the fact that Lemma 1 is not sharp in the corresponding case). So one should avoid $A_\nu \sim A_3$, unless $A_\nu \doteq A_3$ ($\nu = 1, 2$). If that is done our modified estimates remain minimal (assume (39_j)). The corresponding "counterexamples" can be obtained by allowing larger f (near λ_3) or by considering for A functions which vanish near ∞ or behave like $V_i/\lambda_i^{\kappa_i}$.

If (39_i) does not hold, i.e., if V_i increases rather rapidly, then Theorem A_1, for instance, gives no longer a minimal bound. This follows from the following result. (In this case $\lambda_3 = \lambda_1$.)

THEOREM A_1*. *Suppose that* $V_1 > \lambda_1{}^{\Delta}$ *for every* $\Delta > 0$, *and that*

$$0 \leqslant \kappa_1 < \kappa_3 \leqslant 1.$$

Then $A_{\lambda_1}^{\kappa_1} \ll V_1$ *implies* $A_{\lambda_1}^{\kappa_3} \ll V_3 = \lambda_1^{\kappa_3}(V_1/\lambda_1^{\kappa_1})(V_1/A_1V_1')^{\kappa_3-\kappa_1}$. *If* $A_1 \geqslant 1$, $V_1(x) \asymp V_1(x+1)$, *then this is a minimal estimate.*

In order to *prove* this, the integral $A_{\lambda_1}^{\kappa_3}$ is split into two parts:

$$A_{\lambda_1}^{\kappa_3}(x, \tilde{x}) + \int_{\tilde{x}}^{x} (\lambda_1(x) - \lambda_1(t))^{\kappa_3-1}\,\lambda_1'(t)\,A(t)\,dt = I_1 + I_2,$$

$$\lambda_1(\tilde{x}) = \lambda_1(x) - \frac{V_1(x)}{V_1'(x)}\,\lambda_1'(x).$$

Similarly as in the proof of Theorem 2 one shows that $I_1 \leqslant V_3$ (by partial integration) and $I_2 \leqslant V_3$ (by the mean value theorem for integrals). The minimality can be obtained from Lemma 8. There are more changes in the other parts of Theorem 5 if (39_i) does not hold.

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